

THE DISTRIBUTIONS OF THE SECOND
MOMENT AND OF THE CORRELATION
COEFFICIENT IN SAMPLES FROM
POPULATIONS OF TYPE A

BY

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FROM THE STATISTICAL INSTITUTE AT THE UNIVERSITY OF LUND
DIRECTOR: PROF. S. D. WICKSELL

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C. W. K. GLEERUP

LEIPZIG
OTTO HARRASSOWITZ

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PREFACE.

The starting-point for this investigation is derived from one of Professor S. D. Wicksell's works, some formulas in which present traces of a formal connexion between frequency distributions of Charlier's A-Type and Romanowsky's Generalization of Type III. With the kind permission of Professor Wicksell an investigation of this connexion was commenced, in the course of which it appeared that this formal connexion might be applied with advantage to an inquiry into the law of distribution of the error of the second moment.

Now that this work is ready for the press, it is a great pleasure for me to express my sincere gratitude to my teacher and chief, Professor S. D. Wicksell, for the many valuable suggestion as well as for the general assistance and advice he has given me during my investigation.

To my former colleague, now Insurance Actuary in Stockholm, Mr Tage Larsson, I would tender my cordial thanks for his help during my studies as well as for the agreeable collaboration extending over many years.

My obligations are also due to Professor Knut Lundmark for his kindness to include these researches in »Meddelanden från Lunds Astronomiska Observatorium».

Lund in November, 1937.

C.-E. Quensel.

CHAPTER 1.

Introduction.

If a statistical series (containing one or more attributes, *univariate* or *multivariate*) be given, this series can be regarded as part of a larger series (which series in the great majority of cases can be conceived as containing an unlimited number of individuals), or, as it is usually expressed, the statistical series is *a random sample* from an (infinite) *parent population*.

When two or more statistical series are being compared, the question arises whether these different series can be regarded as derived from one and the same parent population, and whether any differences that may exist between the series have only arisen out of chance elements incidental to the limited number in the series.

As a suitable description of a statistical series and its properties, use is made of certain quantities computed from the individual values contained in the series, *characteristics*, and for this purpose recourse has for long been had to the *moments*,¹ or the mean values of the different powers of the individual values in the series. A distinction is made between the *moments about zero*, which are calculated direct from the observed values, and the *central moments* (or the *moments about the arithmetic mean*), the calculation of which is based upon the deviation of the individual observations from the mean of the series. Other characteristics closely related to the moments are also utilized, namely the *half-invariants*, or *seminvariants*. (For the relation between these and the moments, see Chapter 2.)

These moments (seminvariants), like the individual observations, may be regarded as variates.

If more than one random sample (each independent of the other) is drawn from one and the same parent population, the computed values of different moments (seminvariants) will vary from series to series.

The random-sample values of the different characteristics have their own laws of error and with them characteristics belonging to these laws. These laws of

¹ The term »moment» was first used by Professor Karl Pearson and is borrowed from Mechanics.

error and their characteristics will depend upon the law of distribution of the given parent population as well as on the number of individuals contained in the random sample.

Correct analysis of the reliability of the characteristics obtained in a statistical series accordingly demands a knowledge of the laws of error belonging to the characteristics.

The study of these laws of error has consequently become an extremely important part of theoretical statistics. This especially applies, in univariate distributions, to the laws of error for the two first and most important characteristics, the *first moment*, (= *first seminvariant*) or the *mean*, and the *second central moment* (= *second seminvariant*) or the *squared standard deviation*, and, in bivariate and multivariate distributions, also to the laws of error for the *coefficient of correlation* and the *coefficient of regression*.

The study of these laws of error dates mainly from the end of the last century and the beginning of this.

At the outset this study treated of estimations of the *mean errors* of statistical characteristics,¹ and it was laid down that the unknown law of error could be approximately replaced by the *normal law* or *Gaussian law*, which contains only the determined mean error as an arbitrary parameter.

Although this substitution may be regarded as perfectly satisfactory for random samples containing a large number of individuals, this is not the case for small samples.

A powerful impulse was given to the study of the laws of error for random-sample characteristics in small samples by »Student», 1908, in two short but valuable essays.

In the first of these essays² »Student» deals with the deduction, firstly, of the law of error applicable to the *central moment of the second order*, secondly, of the law of error for the ratio between *the deviation of the sample mean from the mean of the parent population* and the *standard deviation* in the random sample, all on the condition that the parent population is distributed according to the normal law.

The closely related law of error for the *second moment about the (unknown) mean of the parent population* was certainly deduced earlier by Helmert,³ but the

¹ L. N. G. Filon and K. Pearson. »On the probable Error of Frequency Constants and on the Influence of random Selection on Variation and Correlation». Phil. Trans. A 191. (1898).

W. F. Sheppard. »On the Application of the Theory of Error to Cases of Normal Distribution and Normal Correlation». Phil. Trans. A 192. (1899).

² »Student». »On the probable Error of a Mean». Biometrika 6. (1908).

Respecting the identity of »Student», see H. Hotelling. »British Statistics and Statisticians to day». Journal of the American Statistical Association. 25. (1930).

³ Helmert. F. R. »Ueber die Wahrscheinlichkeit der Potenzsummen der Beobachtungsfehler und über einige damit in Zusammenhang stehenden Fragen». Zeitschrift für Mathematik und Physik. 21 Jahrg. 1876.

See also Czuber, E. »Theorie der Beobachtungsfehler». Leipzig 1891, and also »Historical

latter's discovery had fallen into almost complete oblivion, and »Student» deduced the law of error for the central moment of the second order independently and by another method. »Student's» deduction started from the statistical characteristics of the normally distributed variate as given and calculated as a function of these, the mathematical expectations of the four first zero-point moments of the sought law of error.

These four moments were subsequently shown to be identical with the corresponding moments of one of Professor Karl Pearson's famous frequency curves, the *Type III curve*, which could accordingly be considered to be the sought law of error. (From the agreement between the four moments it does not however follow, as »Student» also pointed out, that the frequency distribution really is a frequency curve of Type III, though there are good reasons arguing in favour of it.)

At the same time »Student» showed that the first two correlation characteristics between the central moment of the second order and the first moment (the mean) are identical with zero (provided the parent population is normal), and from this he concluded that these two moments could be considered to vary independently of each other.

From these assumptions the distribution of the ratio between the deviation of the sample mean from the mean of the parent population and the standard deviation in the sample was deduced.

Although »Student's» deduction of these important laws cannot be considered entirely satisfactory from a mathematical point of view, these discoveries of his are of great value to theoretical statistics.

»Student's» method of computing — not direct from a given law of distribution as a starting-point, but from its characteristics — the different characteristics of the law of error applicable to the second moment was subsequently applied by others also for the case of a non-normal parent population and likewise for the laws of error for moments of the higher order, but these later workers were not afterwards successful in deducing, as »Student» did, the law of error from the computed characteristics.

Strictly speaking, »Student» cannot be regarded as the first in this respect, for Thiele¹ had several years earlier deduced for the case of a non-normal parent population as well, *inter alia*, the four first seminvariants of the law of error for the second moment, expressed in seminvariants of the parent population.

As there are two different systems of frequency characteristics, the moments and the seminvariants, much duplicate labour has been spent on account of this circumstance.

Note on the Distribution of the Standard Deviation of Samples of any Size drawn from an indefinitely large Normal Population». *Biometrika* 23. 1931.

¹ T. N. Thiele. »Theory of Observations». London 1903. Reprinted in »Annals of Mathematical Statistics» 1931.

The most exhaustive work in this field probably emanates from Tchouproff,¹ who calculated the *moments* of the laws of error not only for the second moment but also for the third and fourth moments (all characteristics expressed in *moments* of the parent population).

Craig and R. A. Fisher² have subsequently worked out the seminvariants for these laws of error, expressed in seminvariants of the parent population.

Even a hasty comparison of these works will show the advantages of the seminvariants over the moments, for the formulas given by Craig and R. A. Fisher are much less complicated than the corresponding formulas in Tchouproff's work.

Although the statistical characteristics of these laws of error have thus been determined (up to terms of the fourth or sixth order), success has not attended the efforts to find from these the sought laws of error.

With reference to the *correlation coefficient* (the mean of the product of the deviations of both variates from their respective means, expressed in the respective standard deviations as units) so important for bivariate distributions, »Student»³ deduced the law of error applicable to this by means of theoretical speculations supported by experimental investigations, though only for the special case of the parent population of the two variates being assumed to be normally distributed and in addition complete independence being assumed between the two variates. The deduced law of error was also in this case one of Pearson's frequency curves, the symmetrical *Type II curve* limited in both directions.

Like the deduction of the law of error for the second moment, this deduction also suffered from a certain incompleteness.

Both the laws of error found by »Student», however, were fully proved a little later by R. A. Fisher,⁴ who demonstrated that in samples from normal bivariate parent populations the moments of the first order and the central

¹ Al. A. Tchouproff. »On the mathematical Expectation of the Moments of frequency Distributions». Part I and II. Biometrika 12. 1918-1919.

Compare also A. E. R. Church. »On the Moments of the Distribution of the squared Standard Deviation for Samples of N drawn from an indefinitely large Population». Biometrika 17. 1925. »On the Means and squared Standard Deviations of small Samples from any Population». Biometrika 18. 1926.

Karl Pearson. »Further Contributions to the Theory of small Samples», Biometrika 17. 1925.

² C. C. Craig. »An Application of Thiele's Seminvariants to the Sampling Problem». Metron 7. 1927-1928.

R. A. Fisher. »Moments and Product Moments of Sampling Distributions». Proceedings of the London Math. Soc. Series 2. 30. 1928.

³ »Student» »The probable Error of a Correlation Coefficient». Biometrika 6. 1908-1909.

⁴ R. A. Fisher. »Frequency Distributions of the Values of the Correlation Coefficient in Samples from an indefinitely large Population». Biometrika 10. 1915.

Compare also »On the Distribution of the Standard Deviation of small Samples». (Editorial) Biometrika 10. 1915.

Karl Pearson »On the Distribution of the Correlation Coefficient in small Samples». Biometrika 11. 1915-1917.

moments of the second order varied quite independently of each other, and who deduced the trivariate law of distribution according to which the three moments of the second order are distributed. »Student's» special law of distribution for the correlation coefficient was shown to be correct and was at the same time given the more general formula for the case of the two variates of the parent population not being independent. This law, however, cannot be represented by means of any previously known frequency curve.

The mode of deduction employed by R. A. Fisher is based on a geometrical view in multidimensional space.

Another method of proving these laws of distribution was given by Romanowsky¹ some years later. Romanowsky made use of the *characteristic function*, first of all deducing the characteristic function of the sought law of error and from that the law of error.

This method does away with the multidimensional geometric operation. Romanowsky starts exclusively from a normal bivariate parent population, and hence naturally obtains the same final result as R. A. Fisher.

On the basis of certain experimental investigations E. S. Pearson² has proved that the law of distribution deduced by »Student» for the relation between the deviation of the sample mean from the mean of the population and the standard deviation is also practically valid for the case in which the parent population does not deviate overmuch from the normal.

But all of the works enumerated here have a normal parent population as their baseline. However valuable the normal distribution may be for the theory of statistics, it is nevertheless only a special case, and one that may never occur in reality. If the parent population deviates from the normal, the laws of error found for the statistical characteristics will no longer be valid, but will have to be interpreted as approximations to the correct expressions and be furnished with correction terms.

If the parent population is not normal, recourse must be had to other frequency curves in order to describe it by means of mathematical expressions. Attempts have been made to deduce the laws of error for some characteristics on the assumption that the parent population can be represented by one of Pearson's frequency curves, but these attempts have been unsuccessful. However, when no knowledge is possessed of the cause of the variation, a frequency distribution of a variate cannot be conceived to be described otherwise than approximately by means of a frequency curve determined only by a few parameters.

In such cases, however, we may describe the frequency distribution (and likewise a multivariate distribution) in the form of an (infinite) series, where the first term, the principal term, consists of the normal law of error, and where

¹ Romanowsky. »On the Moments of Standard Deviations and of Correlation Coefficient in Samples from Normal Population». Metron 5. 1925.

² E. S. Pearson. »The Distribution of Frequency Constants in Small Samples from Non-normal Symmetrical and Skew Populations». Biometrika 21. 1929.

the remaining terms are correction terms whose significance on the whole declines as their order rises. This series is the so-called »Gram-Charlier's A-series» (see Chapter 3).

An attempt to deduce the law of error for the second central moment in the case of a parent population of this type was undertaken by Baker,¹ although his attempt was not a success. For this purpose Baker used the method introduced by R. A. Fisher, but confined his attention to the simplified assumption that the random sample contains only three or four individuals.

In the present work I purpose showing by the method introduced by Romanowsky (although considerably simplified) that simple expressions of the laws of error for the second central moment as well as the correlation coefficient are obtained by starting from a parent population (conceived to be infinitely large) described by means of a »Gram-Charlier's A-series», and that the laws of distribution obtained by »Student» and R. A. Fisher are the first and most important links in these expressions.

¹ G. A. Baker. »Note on the Distribution of the Standard Deviations and Second Moments of Samples from a Gram-Charlier Population». *Annals of Math. Statistics* 6. 1935.

CHAPTER 2.

The Characteristic Function and its Application to the Sampling Problem.

The *characteristic function*, $U(t)$, or as it is also called the *moment generating function*, of a frequency distribution, $F(x)$, is defined by the relation (1)

$$(1) \quad U(t) = \int_{-\infty}^{\infty} e^{xt} F(x) dx,$$

If the parameter t is an imaginary quantity, this integral is always convergent and less than 1, as

$$\int_{-\infty}^{\infty} F(x) dx = 1.$$

Developing e^{xt} into a power series and integrating term by term, we obtain (2)

$$(2) \quad U(t) = 1 + \nu_1' t + \nu_2' \frac{t^2}{2!} + \dots + \nu_n' \frac{t^n}{n!} + \dots,$$

where ν_n' are the different moments about zero.

If the variate x is measured from its mean, we get (2 a)

$$(2 \text{ a}) \quad U(t) = 1 + \nu_2 \frac{t^2}{2!} + \dots + \nu_n \frac{t^n}{n!} + \dots,$$

where ν_n are the moments about the arithmetic mean (the central moments).

If, instead, $\log U(t)$ is developed as a power series, we obtain (3)

$$(3) \quad \log U(t) = 1 + \sum_1^{\infty} \lambda_n \frac{t^n}{n!},$$

where λ_n denote the *seminvariants* defined by Thiele. The name indicates their property of being, except for the first one, independent of the value from which the variate is measured. They have also been given the name of *cumulants* or *cumulative moments* on account of another property they have.

(The relations between moments and seminvariants up to the twelfth order are to be found in Craig's previously cited work.)

For a bivariate distribution $F(x, y)$ the characteristic function can also be defined as

$$(4) \quad U(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1 x + t_2 y} F(x, y) dx dy.$$

Developing $U(t_1, t_2)$ as a power series, we obtain

$$(4 a) \quad U(t_1, t_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\nu_{ij}'}{i! j!} t_1^i t_2^j, \quad [\nu_{00}' = 1]$$

and converting $\log U(t_1, t_2)$ into a power series, we obtain

$$(5) \quad \log U(t_1, t_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda_{ij}}{i! j!} t_1^i t_2^j, \quad [\lambda_{00} = 0].$$

In the event of both the variates being independent of each other, that is

$$F(x, y) = F_1(x) \cdot F_2(y),$$

we obtain

$$U(t_1, t_2) = U_1(t_1) \cdot U_2(t_2)$$

and we have

$$\begin{aligned} \nu_{ij}' &= \nu_{i0}' \cdot \nu_{0j}', \\ \lambda_{ij} &= 0 \text{ if both } i \text{ and } j \neq 0. \end{aligned}$$

The frequency distribution can be obtained from the characteristic function by *Fourier's Inversion Theorem*.

$$(6 a) \quad F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\omega i) e^{-x\omega i} d\omega,$$

$$(6 b) \quad F(x, y) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(\omega_1 i, \omega_2 i) e^{-x\omega_1 i - y\omega_2 i} d\omega_1 d\omega_2.$$

provided the integrals are convergent.

By the use of the characteristic function many theoretical problems receive a simple solution.

If the frequency function, $f(x)$, of the variate x is given, and the frequency function, $F(z)$, of $z = \psi(x)$ is sought, we can first determine the characteristic function of this sought frequency distribution.

$$(7) \quad U(t) = \int_{-\infty}^{\infty} F(z) e^{zt} dz = \int_{-\infty}^{\infty} f(x) e^{\psi(x)t} dx,$$

as

$$F(z) dz = f(x) dx.$$

This method can be extended to functions of two or more variates. The characteristic function of the correlation distribution of $z_1 = g(x, y)$ and $z_2 = h(x, y)$ when the bivariate distribution of x and y is $F(x, y)$ is determined by the relation

$$(8) \quad U(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) e^{g(x, y)t_1 + h(x, y)t_2} dx dy.$$

The characteristic function can also be used in other cases.

If x is a variate belonging to an infinitely large parent population subject to a certain law of distribution $f(x)$, and if a random sample containing N individuals is taken from this population and the values

$$(9 a) \quad z_1 = \varphi(x_1) + \varphi(x_2) + \varphi(x_3) + \dots + \varphi(x_N),$$

$$(9 b) \quad z_2 = \psi(x_1) + \psi(x_2) + \psi(x_3) + \dots + \psi(x_N)$$

are determined, what is then the characteristic function of the distribution of z_1 and z_2 , $F(z_1, z_2)$?

The frequency of a pair of values lying between the limits $z_1, z_1 + dz_1$ and $z_2, z_2 + dz_2$ is given by the relation

$$F(z_1, z_2) dz_1 dz_2 = \int \dots \int f(x_1) f(x_2) \dots f(x_N) dx_1 dx_2 \dots dx_{N-2},$$

where the integral is to be extended over all the combinations of x_i which satisfy the above relations. (9 a, 9 b).

The characteristic function will be

$$\begin{aligned}
 U(t_1, t_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(z_1, z_2) e^{z_1 t_1 + z_2 t_2} dz_1 dz_2 = \\
 (10) \quad &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x_1) f(x_2) \dots f(x_N) e^{t_1[\varphi(x_1) + \varphi(x_2) \dots \varphi(x_N)] + t_2[\psi(x_1) + \dots \psi(x_N)]} dx_1 dx_2 \dots dx_N = \\
 &= \left[\int_{-\infty}^{\infty} f(x) e^{\varphi(x)t_1 + \psi(x)t_2} dx \right]^N.
 \end{aligned}$$

This formula can readily be extended to more than two functions z_1 and z_2 and also to bivariate and multivariate distributions, provided the number of variates z_i is less than the number of individuals N in the random sample.

As a special case let us take that in which $\varphi(x)$ and $\psi(x)$ are different powers of the variate, x^i and x^j . The characteristic function of the distribution of two moments about zero, ν_i' and ν_j' , will thus be given by the relation

$$(11) \quad U(t_1, t_2) = \left[\int_{-\infty}^{\infty} f(x) e^{\frac{x^i}{N} t_1 + \frac{x^j}{N} t_2} dx \right]^N.$$

Similarly, the characteristic function of the distribution of the zero moments $\nu_{ij}', \nu_{kl}', \nu_{mn}' \dots$ in random samples from a bivariate population is obtained by the formula

$$(12) \quad U(t_1, t_2, t_3, \dots) = \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{\frac{x^i y^j}{N} t_1 + \frac{x^k y^l}{N} t_2 + \frac{x^m y^n}{N} t_3 \dots} dx dy \right]^N.$$

CHAPTER 3.

"Gram-Charlier's A-Series" and "Romanowsky's Generalization of Pearson's Type III".

As these two different kinds of frequency curve systems will have considerable application in the following exposition, it seems to me desirable to start with a brief description of these frequency distributions, their characteristics, and some of their properties.

a) "Gram-Charlier's A-Series".

A frequency distribution, $f(x)$, which fulfils certain conditions can be converted into a convergent infinite series of the type ¹

$$(13) \quad f(x) = \varphi(x) + \sum_1^{\infty} (-1)^n \frac{A_n}{n!} \frac{\delta^n \varphi(x)}{\delta x^n},$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi b}} e^{-\frac{1}{2} \frac{(x-a)^2}{b}},$$

and

$$\frac{\delta^n \varphi(x)}{\delta x^n} = \varphi^{(n)}(x) = (-1)^n H_n(x) \varphi(x),$$

in which

$$(14) \quad H_n(x) = \frac{1}{b^{n/2}} \left[\frac{(x-a)^n}{b^{n/2}} - \frac{n(n-1)}{2 \cdot 1!} \frac{(x-a)^{n-2}}{b^{n/2-1}} + \right. \\ \left. + \frac{n(n-1)(n-2)(n-3)}{4 \cdot 2!} \frac{(x-a)^{n-4}}{b^{n/2-2}} - \dots \right]$$

are the well-known Hermitian polynomials.

¹ Regarding these conditions see CHARLIER, C. V. L.: Application de la Théorie des Probabilités à l'Astronomie. Traité du Calcul des Probabilités et ses Applications. Paris, 1931.

This series is the well-known »Gram-Charlier's *A-Series*». The constants A_n are defined by the relation

$$A_n = \int_{-\infty}^{\infty} f(x) H_n(x) dx.$$

If the two parameters in $\varphi(x)$, a and b , are determined so that $a = \lambda_1$, $b = \lambda_2$, that is, identical with the two first seminvariants of the distribution $f(x)$, the two first coefficients vanish, that is $A_1 = A_2 = 0$. To determine these two parameters in this way is by no means necessary, however, even if it is advantageous for practical reasons.

The characteristic function of this distribution becomes

$$(15) \quad U(t) = e^{at + \frac{1}{2}bt^2} \left[1 + \sum_1^{\infty} \frac{A_n}{n!} t^n \right],$$

and the connexion between the seminvariants λ_n of the distribution and the A -coefficients is given by the relation

$$1 + \sum_1^{\infty} \frac{A_n}{n!} t^n = e^{(\lambda_1 - a)t + \frac{\lambda_2 - b}{2!} t^2 + \sum_3^{\infty} \frac{\lambda_n}{n!} t^n}.$$

If a and b are equal to the two first seminvariants, the relations given below are obtained between the twelve first A -Coefficients and the seminvariants.

$$(16) \quad \begin{aligned} A_1 &= 0 \\ A_2 &= 0 \\ A_3 &= \lambda_3 \\ A_4 &= \lambda_4 \\ A_5 &= \lambda_5 \\ A_6 &= \lambda_6 + 10 \lambda_3^2 \\ A_7 &= \lambda_7 + 35 \lambda_4 \lambda_3 \\ A_8 &= \lambda_8 + 56 \lambda_5 \lambda_3 + 35 \lambda_4^2 \\ A_9 &= \lambda_9 + 84 \lambda_6 \lambda_3 + 126 \lambda_5 \lambda_4 + 280 \lambda_3^3 \\ A_{10} &= \lambda_{10} + 120 \lambda_7 \lambda_3 + 210 \lambda_6 \lambda_4 + 126 \lambda_5^2 + 2100 \lambda_4 \lambda_3^2 \\ A_{11} &= \lambda_{11} + 165 \lambda_8 \lambda_3 + 330 \lambda_7 \lambda_4 + 462 \lambda_6 \lambda_5 + 4620 \lambda_5 \lambda_3^2 + 5775 \lambda_4^2 \lambda_3 \\ A_{12} &= \lambda_{12} + 220 \lambda_9 \lambda_3 + 495 \lambda_8 \lambda_4 + 792 \lambda_7 \lambda_5 + 462 \lambda_6^2 + 9240 \lambda_6 \lambda_3^2 + \\ &\quad + 27720 \lambda_5 \lambda_4 \lambda_3 + 5775 \lambda_4^3 + 15400 \lambda_3^4. \end{aligned}$$

These relations, of course, are the same as those between the central moments and the seminvariants, the only difference being that all terms containing the second seminvariant vanish.

However, it is often impracticable to describe a frequency distribution by means of a Gram-Charlier's *A-Series* [even if it is theoretically possible], since in many cases a considerable number of terms must be included in order to present a tolerably correct picture of the frequency distribution by an *A-Series*.

Only if the distribution deviates moderately from the normal distribution $\varphi(x)$ can a development in which only a few terms are included give a fairly correct view of the frequency distribution.

Such a case is presented if the variate x is conceived as being generated by the action of a large number (s) of sources of errors or elementary errors which are independent of each other and which have seminvariants of the same order of magnitude.

Then

$$\lambda_r^{(x)} = \lambda_r^{(1)} + \lambda_r^{(2)} + \lambda_r^{(3)} + \dots + \lambda_r^{(s)} = s l_r$$

if l_r denotes the mean value of the seminvariants of the elementary errors.

The standardized seminvariants

$$\gamma_r = \frac{\lambda_r^{(x)}}{[\lambda_2^{(x)}]^{r/2}} = \frac{l_r}{s^{r/2-1} (l_2)^{r/2}}$$

will then be of the order $s^{1-r/2}$.

Provided the parameters a and b are selected equal to the two first seminvariants, the standardized *A*-coefficients, $a_r = \frac{A_r}{\lambda_2^{r/2}}$, will then be of the following order

$$\begin{aligned} a_3 &\sim \frac{1}{s^{1/2}} \\ a_4, a_6 &\sim \frac{1}{s} \\ a_5, a_7, a_9 &\sim \frac{1}{s^{3/2}} \\ &\vdots \\ a_{3i-4}, a_{3i-2}, a_{3i} &\sim \frac{1}{s^{i/2}}. \\ &\dots\dots\dots \end{aligned}$$

The following two qualities of Hermite's polynomials will be given a short reference here, as they are used in the following exposition.

1) If a change of the zero-point is made in such manner that the variate x is measured from another point ($x = x' + a$), the Hermitian polynomials of the previous variate can be expressed as a series of Hermite's polynomials of the new variate

$$(17) \quad H_n(x) = H_n(x') + \binom{n}{1} \frac{a}{\lambda_2} H_{n-1}(x') + \binom{n}{2} \left(\frac{a}{\lambda_2} \right)^2 H_{n-2}(x') + \dots + \left(\frac{a}{\lambda_2} \right)^n.$$

This formula can easily be verified either algebraically or by derivation of $e^{-\frac{1}{2} \frac{(x'+a)^2}{\lambda_2}}$.

2) $H_n(x)$ includes a parameter λ_2 , that is to say, the polynomials ought more generally to be written ${}_{\lambda_2}H_n(x)$. If we want to pass from polynomials with the parameter L_2 to polynomials with the parameter λ_2 , the following relation can be put down.

$$(18) \quad {}_{\lambda_2}H_n(x) = \left(\frac{L_2}{\lambda_2} \right)^{n/2} \left[{}_{L_2}H_n(x) - \frac{n(n-1)}{2} \frac{\lambda_2 - L_2}{L_2 \lambda_2} {}_{L_2}H_{n-2}(x) + \right. \\ \left. + \frac{n(n-1)(n-2)(n-3)}{4 \cdot 2!} \left(\frac{\lambda_2 - L_2}{L_2 \lambda_2} \right)^2 {}_{L_2}H_{n-4}(x) - \dots \right].$$

We always have

$$\int_{-\infty}^{\infty} {}_{\lambda_2}H_n(x) {}_{\lambda_2}\varphi(x) dx = 0, \quad n \neq 0,$$

but, according to formula 18, we shall find

$$(19) \quad \int_{-\infty}^{\infty} {}_{\lambda_2}H_{2n}(x) {}_{L_2}\varphi(x) dx = (-1)^n \frac{2n!}{2^n n!} \cdot \left[\frac{\lambda_2 - L_2}{\lambda_2^2} \right]^n;$$

$$\int_{-\infty}^{\infty} {}_{\lambda_2}H_{2n+1}(x) {}_{L_2}\varphi(x) dx = 0,$$

formulas that are reminiscent of the formulas for the central moments of the normal distribution

$$\nu_{2n} = \frac{2n!}{2^n n!} \lambda_2^n; \quad \nu_{2n+1} = 0.$$

b) *Pearson's Type III* and *Romanowsky's Generalization of Type III*.

The frequency curve that is generally called »Pearson's Type III» or shortly »Type III» is defined by the formula (20)

$$(20) \quad F(z) = \frac{(z - a)^{b-1} e^{-\frac{z-a}{\gamma}}}{\Gamma(b) \gamma^b}.$$

Contrary to the normal curve this frequency distribution is not unlimited in both directions, but is characterized by the fact that there exists a lower limit for the variate but no upper limit.

In the formula (20) a denotes this lower limit value. If the variate z is measured from this point, the formula (20) is changed into (20 a), which special formula will constantly be used below.

$$(20 \text{ a}) \quad F(z) = \frac{z^{b-1} e^{-\frac{z}{\gamma}}}{\Gamma(b) \gamma^b},$$

$F(z)$ is a function that contains two parameters, b and γ , and should therefore rightly be written $F_{b,\gamma}(z)$. For convenience in notation consideration will here be had only to the parameter b , and the expression $f_b(z)$ will be used to denote a »Type III» of the form (20 a).

The characteristic function of $f_b(z)$ is (21)

$$(21) \quad U(t) = [1 - \gamma t]^{-b}.$$

The seminvariants of the »Type III» distribution will therefore be expressed by the relation (22)¹

$$(22) \quad \sum \frac{\lambda_n}{n!} t^n = -b \log [1 - \gamma t], \quad \lambda_n = b \cdot (n-1)! \gamma^n,$$

*Romanowsky's generalization of Type III*² is a series development in which the first term represents a »Type III» function and the following terms are the derivatives, not of this »Type III» function, but of other »Type III» functions in which the constant b is increased by as many units as the order of the derivatives indicates and in which the constant γ remains unchanged.

$$(23) \quad F(z) = f_b(z) + \sum (-1)^n \frac{a_n}{n!} f_{b+n}^{(n)}(z),$$

¹ J. F. STEFFENSEN, Matematisk Iagttagelselære. Copenhagen 1923.

² W. ROMANOWSKY. Generalisations of some Types of the Frequency Curves of Professor PEARSON. Biometrika 16. 1924.

(In this formula and accordingly in the following $f_b(z)$ and $f_{b+n}(z)$ must, of course, be conceived as having the same value of the parameter γ .)

The function $f_{b+n}^{(n)}(z)$ can be written as a product of the function $f_b(z)$ and a polynomial of the degree n , ${}_bL_n(z)$.

$$(24) \quad f_{b+n}^{(n)}(z) = (-1)^n {}_bL_n(z) f_b(z).$$

The general expression of ${}_bL_n(z)$ is

$$(25) \quad {}_bL_n(z) = \frac{\Gamma(b)}{\gamma^n \Gamma(b+n)} \left[\frac{z^n}{\gamma^n} - \binom{n}{1} (b+n-1) \frac{z^{n-1}}{\gamma^{n-1}} + \right. \\ \left. + \binom{n}{2} (b+n-1)(b+n-2) \frac{z^{n-2}}{\gamma^{n-2}} - \dots \right].$$

The function $f_{b+n}^{(n)}(z)$ may also be written

$$(24 \text{ a}) \quad f_{b+n}^{(n)}(z) = \frac{(-1)^n}{\gamma^n} \left[f_{b+n}(z) - n f_{b+n-1}(z) + \binom{n}{2} f_{b+n-2}(z) - \dots (-1)^n f_b(z) \right]$$

Romanowsky's generalization is in many respects comparable with the »A-Series» but has not received the attention it seems to me to deserve.

The characteristic function of (23) is given by the formula (26)

$$(26) \quad U(t) = [1 - \gamma t]^{-b} \left[1 + \sum \frac{a_n}{n!} \left(\frac{t}{1 - \gamma t} \right)^n \right].$$

In a treatise not yet published Professor Wicksell has introduced certain new frequency characteristics which, when frequency distributions of »Type III» and »Type III Series» are concerned, have in many respects the same qualities as the seminvariants in normal distributions and »A-Series». These characteristics ϑ_n stand in the following relation to the seminvariants

$$(27 \text{ a}) \quad \sum \frac{\lambda_n}{n!} t^n = \sum \frac{\vartheta_n}{n!} \left(\frac{t}{1 - \gamma t} \right)^n, \text{ which gives}$$

$$\frac{\lambda_n}{n!} = \frac{\vartheta_n}{n!} + \binom{n-1}{1} \gamma \frac{\vartheta_{n-1}}{(n-1)!} + \dots + \gamma^{n-1} \vartheta_1,$$

or inversely

$$(27 \text{ b}) \quad \sum \frac{\vartheta_n}{n!} \tau^n = \sum \frac{\lambda_n}{n!} \left(\frac{\tau}{1 + \gamma \tau} \right)^n, \text{ which gives}$$

$$\frac{\vartheta_n}{n!} = \frac{\lambda_n}{n!} - \binom{n-1}{1} \gamma \frac{\lambda_{n-1}}{(n-1)!} + \binom{n-1}{2} \gamma^2 \frac{\lambda_{n-2}}{(n-2)!} - \dots (-1)^n \gamma^{n-1} \lambda_1.$$

In distributions of »Type III» these constants assume the values (28)

$$(28) \quad \vartheta_{0n} = (-1)^{n-1} b \gamma^n (n-1)!.$$

The relations between these characteristics and the coefficients in the »Type III Series» are given by the formula (29)

$$(29) \quad e^{\sum \frac{\vartheta_n - \vartheta_{0n}}{n!} \tau^n} = 1 + \sum \frac{a_n}{n!} \tau^n, \text{ where } \tau \text{ is substituted for } \left(\frac{t}{1 - \gamma t} \right)$$

and where ϑ_{0n} denote the characteristics that correspond to the main term in »Romanowsky's Type III Series».

The relations between $\vartheta_n - \vartheta_{0n}$ and the coefficients a_n are thus the same as between the seminvariants and the moments.

As in the following exposition these characteristics will serve only as a connecting link between the seminvariants and the coefficients in »Romanowsky's Type III Series» their qualities need not to be discussed in detail here.

There are, however, two qualities that should be mentioned.

If several variates, independent of each other, are distributed according to the same or to different »Type III Series» with the *same* value of the constant γ , the characteristics ϑ_n belonging to the distribution of the sum $z = z_1 + z_2 + \dots + z_N$ are equal to the sum of the characteristics of the different variates

$$\vartheta_n = {}_1\vartheta_n + {}_2\vartheta_n + \dots + {}_N\vartheta_n.$$

Thus in this special case they have the cumulative property of the seminvariants, and I therefore suggest the name »*Type III Cumulants*».

It should be observed, however, that this cumulative quality is valid for addition only. Should one or more of the variates z_i be included with a negative sign in the sum, the formula is not valid.

Another important quality of these characteristics is that they are independent of each other in that their sampling values are uncorrelated with each other, if samples are taken from a »Type III» distribution with the value of the lower limit a and the value of the constant γ known.

This can be demonstrated in the following way, which is, however, not quite sufficient, since it merely proves that the first correlation moments between them are 0.

If a sample of N individuals is taken from a »Type III» distribution characterized by the constants b and γ , and if the different seminvariants λ_n and the different characteristics ϑ_n are calculated, the mathematical expectations are attained in case terms of the magnitude $1/N$ and lower are omitted.

$$\begin{aligned} m(\lambda_n) &= b\gamma^n (n-1)! \\ m(\vartheta_n) &= b\gamma^n (-1)^n (n-1)! \end{aligned}$$

It must be observed that for the calculation of ϑ_n from the observations the values of the lower limit a and the constant γ must be known, although no knowledge of the third constant b is required. According to formula (27 a) we have

$$\sum \frac{\vartheta_n}{n!} \left(\frac{t}{1-\gamma t} \right)^n = \sum \frac{\lambda_n}{n!} t^n.$$

If we regard the errors of the sampling values as differentials, we have

$$\begin{aligned} \sum \frac{\delta \vartheta_i}{i!} \left(\frac{t_1}{1-\gamma t_1} \right)^i &= \sum \frac{\delta \lambda_i}{i!} t_1^i \\ \sum \frac{\delta \vartheta_j}{j!} \left(\frac{t_2}{1-\gamma t_2} \right)^j &= \sum \frac{\delta \lambda_j}{j!} t_2^j. \end{aligned}$$

Multiplying these two expressions by each other and taking the mean values of the different products, we get

$$\sum \sum \frac{k(\vartheta_i \vartheta_j)}{i! j!} \left(\frac{t_1}{1-\gamma t_1} \right)^i \left(\frac{t_2}{1-\gamma t_2} \right)^j = \sum \sum \frac{k(\lambda_i \lambda_j)}{i! j!} t_1^i t_2^j,$$

where $k(x_i x_j)$ denotes the correlation moment between the errors in x_i and x_j .

According to a formula given by St Georgescu,¹ however,

$$\sum \sum \frac{k(\lambda_i \lambda_j)}{i! j!} t_1^i t_2^j = \frac{1}{N} e^{\sum \frac{\lambda_n}{n!} [(t_1+t_2)^n - t_1^n - t_2^n]} = \frac{1}{N} \frac{U(t_1+t_2)}{U(t_1)U(t_2)}.$$

From this expression we get

$$\sum \sum \frac{k(\vartheta_i \vartheta_j)}{i! j!} \left(\frac{t_1}{1-\gamma t_1} \right)^i \left(\frac{t_2}{1-\gamma t_2} \right)^j = \frac{1}{N} \frac{(1-\gamma t_1)^b (1-\gamma t_2)^b}{[1-\gamma(t_1+t_2)]^b}.$$

¹ St Georgescu. Further Contributions to the Sampling Problem. Biometrika 24. 1932.

If we put

$$\frac{t_1}{1 - \gamma t_1} = \tau_1, \quad \frac{t_2}{1 - \gamma t_2} = \tau_2$$

we get

$$\sum \sum \frac{k(\vartheta_i \vartheta_j)}{i! j!} \tau_1^i \tau_2^j = \frac{1}{N} \left(\frac{1}{1 - \gamma^2 \tau_1 \tau_2} \right)^b,$$

From this formula it is at once evident that $k(\vartheta_i \vartheta_j)$, $(i \neq j)$, the first correlation moment, is 0.

I refrain, however, from giving in this connection a proof that the higher correlation characteristics are also 0 and from treating the more important case of in what degree these characteristics are independent of each other when the constant γ is determined from the observations and not known before.

CHAPTER 4.

The Distribution of the Second Moment in Samples from Normal Parent Populations.

a) The Distribution of the Square of the Variate.

We here consider the case that the variate x , which hereafter will always be considered as measured from the arithmetic mean of the parent population, is distributed in accordance with the normal curve of error:

$$\varphi(x) = \frac{1}{\sqrt{2\pi\lambda_2}} e^{-\frac{1}{2}\frac{x^2}{\lambda_2}}.$$

A simple transformation gives the distribution of the square of the variate. If $z = \varphi(x)$ the distribution $F(z)$ of z is arrived at by the general formula $F(z)dz = \varphi(x)dx$. In this case, where $z = x^2$, it must, however, be observed that only *one* value of z corresponds to the *two* values of x with the same value but opposite signs, and therefore the formula will be

$$(30) \quad F(z)dz = [\varphi(x) + \varphi(-x)]dx.$$

This formula results in

$$F(z) = \frac{1}{\sqrt{2\pi\lambda_2}} z^{-\frac{1}{2}} e^{-\frac{1}{2}\frac{z}{\lambda_2}}.$$

Thus x^2 is distributed according to a »Pearson's Type III», with the value of the lower limit $= 0$ and with the parameters $b = \frac{1}{2}$ and $\gamma = 2\lambda_2$. Or, if the notation from Chapter 3 is used,

$$F(z) = f_{\frac{1}{2}}(z).$$

b) **The Distribution of the Second Moment about the Mean of the Parent Population.**

The distribution of the second moment about zero

$$z = \frac{1}{N} [x_1^2 + x_2^2 + x_3^2 + \dots + x_N^2]$$

is most conveniently obtained by first determining the corresponding characteristic function (according to formula (11))

$$(31) \quad U(t) = \left[\int_{-\infty}^{\infty} \varphi(x) e^{\frac{x^2}{N} t} dx \right]^N = \left[1 - \frac{2\lambda_2}{N} t \right]^{-\frac{N}{2}}.$$

This, however, is nothing else but the characteristic function of a frequency distribution of »Pearson's Type III» with the parameters $b = \frac{N}{2}$ and $\gamma = \frac{2\lambda_2}{N}$.

$$(32) \quad F(z) = \frac{1}{\Gamma\left(\frac{N}{2}\right)} \left[\frac{N}{2\lambda_2} \right]^{\frac{N}{2}} z^{\frac{N}{2}-1} e^{-\frac{N}{2\lambda_2} z},$$

or in the notation from *Chapter 3*

$$(32 \text{ a}) \quad F(z) = f_{\frac{N}{2}}(z).$$

This simple method of determining the distribution of the second moment by aid of the characteristic function was first used by S. D. Wicksell in his university lectures. (See also Craig¹). The seminvariants L_n of this distribution are according to formula (22) in *Chapter 3* given by the equation

$$L_n = (n-1)! \left(\frac{2}{N} \right)^{n-1} \lambda_2^n.$$

c) **The Distribution of the Second Moment about the Mean of the Sample.**

If from the distribution of the second moment about the mean of the *parent population* we desire to pass over to the distribution of the second moment about

¹ C. C. CRAIG. An Application of Thiele's Seminvariants to the Sampling Problem. *Metron*. Vol. 7. 1927—1928.

the mean of the *sample*, it seems to be most suitable first to study the correlation distribution of the sampling values of the first and second moment about the mean of the parent population. The two marginal distributions of this correlation distribution are known, one belonging to the normal type and the other to »Type III». Here we have a correlation distribution of a type that has been studied by S. D. Wicksell.¹

The characteristic function $U(t_1, t_2)$ of the correlation distribution $F(z_1, z_2)$ of the two moments

$$z_1 = \frac{1}{N} [x_1^2 + x_2^2 + x_3^2 + \dots + x_N^2]$$

$$z_2 = \frac{1}{N} [x_1 + x_2 + x_3 + \dots + x_N]$$

is according to formula (11) given by the formula

$$(33) \quad U(t_1, t_2) = \left[\int_{-\infty}^{\infty} \varphi(x) e^{\frac{x^2}{N} t_1 + \frac{x}{N} t_2} dx \right]^N = \left[1 - \frac{2\lambda_2}{N} t_1 \right]^{-\frac{N}{2} \frac{\lambda_2}{2N} \frac{t_2^2}{1 - \frac{2\lambda_2}{N} t_1}},$$

which can also be written

$$(33 \text{ a}) \quad U(t_1, t_2) = \left[1 - \frac{2\lambda_2}{N} t_1 \right]^{-\frac{N}{2} \frac{\lambda_2}{2N} t_2^2} e^{\frac{\lambda_2^2 t_1 t_2^2}{e^{N^2} \left[1 - \frac{2\lambda_2}{N} t_1 \right]}}.$$

We observe that $U(t_1, t_2)$ can be written as the product of three factors, the first and the second of which are the characteristic functions of the marginal distributions of z_1 and z_2 , and that the third factor must be the part that determines the correlation between the first and the second moment about the mean of the parent population.

The seminvariants of this correlation surface will be

$$L_{n0} = (n-1)! \left(\frac{2}{N} \right)^{n-1} \lambda_2^n$$

$$L_{02} = \frac{\lambda_2}{N}$$

$$L_{n2} = n! \left(\frac{2}{N} \right)^n \frac{1}{N} \cdot \lambda_2^{n+1}$$

$$L_{0n} = L_{mn} = 0 \text{ if } n \neq 2.$$

¹ S. D. WICKSELL. On Correlation Functions of Type III. *Biometrika* 25. 1933.

If the third factor in $U(t_1, t_2)$ is developed as an infinite series, formula (34) is obtained

$$(34) \quad U(t_1, t_2) = U(t_1) U(t_2) \left[1 + \sum_1^{\infty} \frac{1}{n!} \frac{\lambda_2^n}{N^{2n}} t_2^{2n} \left(\frac{t_1}{1 - \frac{2\lambda_2}{N} t_1} \right)^n \right].$$

From this expression the correlation distribution can be obtained as an infinite series by aid of the »Inversion Theorem

$$F(z_1, z_2) = \left(\frac{1}{2\pi} \right)^2 \int \int U(\omega_1 i, \omega_2 i) e^{-z_1 \omega_1 i - z_2 \omega_2 i} d\omega_1 d\omega_2.$$

If the infinite series is integrated term by term, formula (35) is obtained

$$(35) \quad F(z_1, z_2) = f_{\frac{N}{2}}(z_1) \varphi_{\frac{\lambda_2}{N}}(z_2) + \sum_1^{\infty} (-1)^n \frac{1}{n!} \frac{\lambda_2^n}{N^{2n}} f_{\frac{N}{2}+n}^{(n)}(z_1) \varphi_{\frac{\lambda_2}{N}}^{(2n)}(z_2).$$

A similar correlation distribution has already been given by Wicksell. Wicksell started from a two-dimensional normal correlation surface and investigated the correlation distribution between the mean of one variate and the second moment about the mean of the parent population of the other variate. There exists no difference worth mentioning between the correlation surface deduced by him and the one given above. If in Wicksells formula σ_x is made $= \sigma_y = \sqrt{\lambda_2}$ and $r=1$, that is, if the two variates are supposed to be identical, the above formula is arrived at.

The correlation surface given above is a combination in two dimensions of a »Gram-Charlier's A-Series» and a »Romanowsky's Type III Series».

This correlation distribution is characterized by a strictly parabolic regression.

The regression of the mean of z_1 on z_2 , $m_{z_1}(z_2)$, is most readily obtained by the following relation given by Wicksell.¹

$$(36) \quad m_{z_1}(z_2) = \frac{1}{\varphi(z_2)} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-z_2 \omega_2 i} \left[\frac{\delta U(t_1, \omega_2 i)}{\delta t_1} \right]_{t_1=0} d\omega_2.$$

Now

$$\left[\frac{\delta U(t_1, t_2)}{\delta t_1} \right]_{t_1=0} = \lambda_2 e^{\frac{\lambda_2 t_2^2}{2N}} + \frac{\lambda_2^2}{N^2} t_2^2 e^{\frac{\lambda_2}{2N} t_2^2},$$

¹ S. D. WICKSELL. Analytical Theory of Regression. Meddelande från Lunds Astronomiska Observatorium Ser. II. n:r 69.

and hence we obtain

$$(37) \quad m_{z_1}(z_2) = \lambda_2 + \frac{\lambda_2^2}{N^2} H_2(z_2) = \frac{N-1}{N} \lambda_2 + z_2^2.$$

This correlation distribution (35) can be compared with the normal correlation distribution, which, fully analogously, can be written as an infinite series

$$\varphi(x, y) = \varphi(x) \varphi(y) + \sum \frac{1}{n!} \lambda_{11}^n \varphi^{(n)}(x) \varphi^{(n)}(y).$$

Just as this infinite series can be changed into a definite expression, the normal correlation surface, which can also be written in the following way as the product of two independent frequency functions

$$\varphi(x, y) = \frac{1}{\sqrt{2\pi\lambda_{02}}} e^{-\frac{y^2}{2\lambda_{02}}} \cdot \frac{1}{\sqrt{2\pi\lambda_{20}(1-r^2)}} e^{-\frac{1}{2(1-r^2)} \left[\frac{x}{\lambda_{20}^{1/2}} - r \frac{y}{\lambda_{02}^{1/2}} \right]^2},$$

so can series (35) be changed into a definite expression.

From the »Inversion Theorem» we obtain

$$F(z_1, z_2) = \left(\frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-z_1 \omega_1 i - z_2 \omega_2 i} \left[1 - \frac{2\lambda_2}{N} \omega_1 i \right]^{-\frac{N}{2} \frac{\lambda_2 (\omega_2 i)^2}{1 - \frac{2\lambda_2}{N} \omega_1 i}} d\omega_1 d\omega_2$$

or

$$F(z_1, z_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_1 e^{-z_1 \omega_1 i} \left[1 - \frac{2\lambda_2}{N} \omega_1 i \right]^{-\frac{N}{2}} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-z_2 \omega_2 i} e^{\frac{\lambda_2 (\omega_2 i)^2}{1 - \frac{2\lambda_2}{N} \omega_1 i}} d\omega_2 \right].$$

The last integral can be regarded as a normal frequency function with the second seminvariant $= \frac{\lambda_2}{N \left[1 - \frac{2\lambda_2}{N} \omega_1 i \right]}$, that is

$$\sqrt{\frac{N \left[1 - \frac{2\lambda_2}{N} \omega_1 i \right]}{2\pi\lambda_2}} e^{-\frac{N \left[1 - \frac{2\lambda_2}{N} \omega_1 i \right]}{2\lambda_2} z_2^2}.$$

Thus we get

$$(38) \quad F(z_1, z_2) = \sqrt{\frac{N}{2\pi\lambda_2}} e^{-\frac{N z_2^2}{2\lambda_2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(1 - \frac{2\lambda_2}{N} \omega_1 i \right)^{-\frac{N-1}{2}} e^{-(z_1 - z_2^2) \omega_1 i} d\omega_1.$$

But as $z_1 - z_2^2$ is the second moment about the arithmetic mean of the sample $= \nu_2$, the last integral becomes the frequency function of ν_2

$$(39) \quad F(\nu_2, z_2) = \sqrt{\frac{N}{2\pi\lambda_2}} e^{-\frac{N z_2^2}{2\lambda_2}} \cdot \frac{\left(\frac{N}{2\lambda_2}\right)^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \nu_2^{\frac{N-1}{2}-1} e^{-\frac{N}{2} \frac{\nu_2}{\lambda_2}}.$$

This formula shows with full evidence that in samples from normal parent populations the mean and the second central moment are quite independent of each other.

The frequency distribution of ν_2 becomes

$$F(\nu_2) = \frac{\left(\frac{N}{2\lambda_2}\right)^{\frac{N-1}{2}} \nu_2^{\frac{N-1}{2}-1}}{\Gamma\left(\frac{N-1}{2}\right)} e^{-\frac{N}{2} \frac{\nu_2}{\lambda_2}} = f_{\frac{N-1}{2}}(\nu_2).$$

The seminvariants are given by the formula

$$L_n = \frac{N-1}{2} (n-1)! \left(\frac{2}{N}\right)^n \lambda_2^n.$$

The only difference between these formulas regarding the second central moment and the corresponding formulas regarding the second zero moment is obviously that the parameter b has been changed from $\frac{N}{2}$ into $\frac{N-1}{2}$.

Although these results are well known, this rather short method of demonstrating the independence of the mean and the second moment and of determining the frequency distribution of the second central moment seems to be new. As will be shown in the following chapters, this method will also be of great use when the parent population is not normally distributed.

CHAPTER 5.

The Distribution of the Second Moment in Samples from Frequency Distributions of Type A.

a) The Distribution of the Square of the Variate.

Let us assume the variate to be distributed according to a »Gram-Charlier's A-Series«. Although such a series should be conceived as containing an infinite number of terms, I shall confine the following investigation to the case of a series containing only a finite number of terms.

Thus we consider the variate as being distributed according to the formula

$$F(x) = \varphi(x) + \sum_3^n (-1)^n \frac{A^n}{n!} \varphi^{(n)}(x); \quad \varphi(x) = \frac{1}{\sqrt{2\pi\lambda_2}} e^{-\frac{1}{2} \frac{x^2}{\lambda_2}},$$

or

$$F(x) = \varphi(x) \left[1 + \sum_3^n \frac{A_n}{n!} H_n(x) \right].$$

The distribution of the square of the variate $z = x^2$ is obtained, in this case too, by a transformation according to the formula (30) in *Chapter 4*. As $H_{2n+1}(x)$ is an odd function of x , all terms with odd number will vanish.

$$F(z) dz = 2\varphi(x) \left[1 + \sum \frac{A_{2n}}{2n!} H_{2n}(x) \right] dx.$$

The Hermitian polynomials of even order are polynomials of x^2 and hence become, with regard to $z = x^2$, polynomials $P_n(z)$ of the n th degree.

$$P_n(z) = \frac{1}{\lambda_2^n} \left[\frac{z^n}{\lambda_2^n} - \frac{2n(2n-1)}{2 \cdot 1!} \frac{z^{n-1}}{\lambda_2^{n-1}} + \right. \\ \left. + \frac{2n(2n-1)(2n-2)(2n-3)}{2^2 2!} \frac{z^{n-2}}{\lambda_2^{n-2}} - \dots + (-1)^n \frac{2n!}{2^n n!} \right].$$

Thus we get

$$F(z) = f_{\frac{1}{2}}(z) \left[1 + \sum \frac{A_{2n}}{2n!} P_n(z) \right],$$

where $f_{\frac{1}{2}}(z)$ represents, in accordance with *Chapter 4*, a »Pearson's Type III» with $b = \frac{1}{2}$ and where $\gamma = 2\lambda_2$.

If we compare these polynomials $P_n(z)$ with the polynomials ${}_bL_n(z)$ described in *Chapter 3* formula (25), we find that, ignoring a constant factor, $P_n(z) = {}_bL_n(z)$, if in ${}_bL_n(z)$ we put $b = \frac{1}{2}$ and $\gamma = 2\lambda_2$.

The constant factor is $\frac{2^{2n}\Gamma(n + \frac{1}{2})}{\Gamma(\frac{1}{2})} = \frac{2n!}{n!}$.

Thus we obtain

$$P_n(z) = \frac{2n!}{n!} {}_{\frac{1}{2}}L_n(z)$$

and

$$F(z) = f_{\frac{1}{2}}(z) \left[1 + \sum \frac{A_{2n}}{n!} {}_{\frac{1}{2}}L_n(z) \right]$$

or

$$(40) \quad F(z) = f_{\frac{1}{2}}(z) + \sum (-1)^n \frac{A_{2n}}{n!} f_{\frac{1}{2}+n}^{(n)}(z).$$

If the variate is distributed according to a »Gram-Charlier's A-Series», the square of the variate is distributed according to a »Romanowsky's Generalization of Type III» and the coefficients of the different derivatives in the latter are the coefficients of the derivatives of even number in the former.

The distribution of $z = x^2$ can also be obtained by aid of the characteristic function. Although the result will be the same, this method is more expedient, because it will here, too, be easier to pass over to the distribution of

$$z = \frac{1}{N} \left[x_1^2 + x_2^2 + x_3^2 + \dots + x_N^2 \right]$$

or the second moment about the mean of the parent population.

The characteristic function is given by the formula

$$U(t) = \int_{-\infty}^{\infty} e^{xt} F(x) dx = \int_{-\infty}^{\infty} e^{xt} \left[\varphi(x) + \sum (-1)^n \frac{A_n}{n!} \varphi^{(n)}(x) \right] dx.$$

But according to formula (31)

$$\int_{-\infty}^{\infty} e^{xt} \varphi(x) dx = [1 - 2\lambda_2 t]^{-\frac{1}{2}}$$

and

$$\int_{-\infty}^{\infty} e^{x^2 t} \varphi^{(2n+1)}(x) dx = 0$$

$$\int_{-\infty}^{\infty} e^{x^2 t} \varphi^{(2n)}(x) dx = \frac{1}{\sqrt{2\pi\lambda_2}} \int_{-\infty}^{\infty} H_{2n}(x) e^{-\frac{x^2}{2} \left[\frac{1-2\lambda_2 t}{\lambda_2} \right]} dx = \frac{2n!}{n!} t^n (1 - 2\lambda_2 t)^{-\frac{1}{2}-n}.$$

The last formula is obtained immediately from formula (19) in *Chapter 3*.
Thus we obtain

$$(41) \quad U(t) = [1 - 2\lambda_2 t]^{-1/2} \left[1 + \sum \frac{A_{2n}}{n!} \left(\frac{t}{1 - 2\lambda_2 t} \right)^n \right],$$

which is evidently the characteristic function of the frequency function arrived at before.

The seminvariants L_n of the distribution $F(z)$ are most simply obtained from the characteristic function.

Therefore it seems to be most suitable to use the characteristics ϑ_n defined in *Chapter 3*.

$$e \sum \frac{\vartheta_n}{n!} \left(\frac{t}{1 - 2\lambda_2 t} \right)^n = (1 - 2\lambda_2 t)^{-\frac{1}{2}} \left[1 + \sum \frac{A_{2n}}{n!} \left(\frac{t}{1 - 2\lambda_2 t} \right)^n \right]$$

or, putting

$$e \sum \frac{\vartheta_{0n}}{n!} \left(\frac{t}{1 - 2\lambda_2 t} \right)^n = [1 - 2\lambda_2 t]^{-\frac{1}{2}},$$

$$\vartheta_{0n} = (-1)^{n-1} (n-1)! 2^{n-1} \lambda_2^n,$$

we find

$$e \sum \frac{\vartheta_n - \vartheta_{0n}}{n!} t^n = 1 + \sum \frac{A_{2n}}{n!} t^n.$$

If we express these constants in terms of the seminvariants of the original variate, we obtain the following expression of the first six characteristics.

$$(42) \quad \begin{aligned} \vartheta_1 - \vartheta_{01} &= 0. \\ \vartheta_2 - \vartheta_{02} &= \lambda_4. \\ \vartheta_3 - \vartheta_{03} &= \lambda_6 + 10 \lambda_3^2. \\ \vartheta_4 - \vartheta_{04} &= \lambda_8 + 56 \lambda_5 \lambda_3 + 32 \lambda_4^2. \\ \vartheta_5 - \vartheta_{05} &= \lambda_{10} + 120 \lambda_7 \lambda_3 + 200 \lambda_6 \lambda_4 + 126 \lambda_5^2 + 2000 \lambda_4 \lambda_3^2. \\ \vartheta_6 - \vartheta_{06} &= \lambda_{12} + 220 \lambda_9 \lambda_3 + 480 \lambda_8 \lambda_4 + 792 \lambda_7 \lambda_5 + 452 \lambda_6^2 + \\ &\quad + 9040 \lambda_6 \lambda_3^2 + 26880 \lambda_5 \lambda_4 \lambda_3 + 5280 \lambda_4^3 + 14400 \lambda_3^4. \end{aligned}$$

It must be observed that these expressions do not involve the seminvariant λ_2 . From these expressions it is easy to derive the formulas for the seminvariants. The formulas for the first five seminvariants have been given before by Craig in the treatise mentioned above.

The expression for the sixth seminvariant is

$$(43) \quad \begin{aligned} L_6 = & \lambda_{12} + 60 \lambda_{10} \lambda_2 + 220 \lambda_9 \lambda_3 + 480 \lambda_8 \lambda_4 + 792 \lambda_7 \lambda_5 + 452 \lambda_6^2 + 1200 \lambda_8 \lambda_2^2 + \\ & + 7200 \lambda_7 \lambda_3 \lambda_2 + 12000 \lambda_6 \lambda_4 \lambda_2 + 7560 \lambda_5^2 \lambda_2 + 9040 \lambda_6 \lambda_3^2 + 26880 \lambda_5 \lambda_4 \lambda_3 + \\ & + 5280 \lambda_4^3 + 9600 \lambda_6 \lambda_2^3 + 67200 \lambda_5 \lambda_3 \lambda_2^2 + 38400 \lambda_4^2 \lambda_2^2 + 120000 \lambda_4 \lambda_3^2 \lambda_2 + \\ & + 14400 \lambda_3^4 + 28800 \lambda_4 \lambda_2^4 + 96000 \lambda_3^2 \lambda_2^3 + 3840 \lambda_2^6, \end{aligned}$$

a rather complicated expression, especially in comparison with ϑ_6 (formula 42).

b) The Distribution of the Second Moment about the Mean of the Parent Population.

The characteristic function of

$$z = \frac{1}{N} \left[x_1^2 + x_2^2 + x_3^2 + \dots + x_N^2 \right]$$

becomes

$$U(t) = \left[1 - \frac{2\lambda_2}{N} t \right]^{-\frac{N}{2}} \left[1 + \sum \frac{B_n}{n!} \left(-\frac{t}{1 - \frac{2\lambda_2}{N} t} \right)^n \right],$$

where the coefficients B_n are obtained from the A_n coefficients by the relation

$$1 + \sum \frac{B_n}{n!} t^n = \left[1 + \sum \frac{A_{2n}}{N^n n!} t^n \right]^N.$$

The distribution of z becomes a »Romanowsky's Series of Type III» with

$$b = \frac{N}{2} \text{ and } \gamma = \frac{2\lambda_2}{N}.$$

$$(44) \quad F(z) = f_{\frac{N}{2}}(z) + \sum (-1)^n \frac{B_n}{n!} f_{\frac{N}{2}+n}^{(n)}(z).$$

The characteristics ϑ_n and the seminvariants L_n are obtained from the expressions given before (formulas 42 and 43) of the corresponding characteristics of the distribution of x^2 on multiplication by $N^{-(n-1)}$.

This may seem strange with regard to the relation between ϑ_n and L_n , but it should be noted that in this relation γ is changed from $2\lambda_2$ into $\frac{2\lambda_2}{N}$.

Thus the distribution obtained of the second moment is a »Romanowsky's Generalization of Type III».

c) The Distribution of the Second Moment about the Mean of the Sample.

As is the case when the parent population is normally distributed, it seems to me that the best way of determining the distribution formula of the second moment about the mean of the sample is first of all to derive the correlation distribution of the first and the second moment about the mean of the parent population and from that point to pass on to the distribution of the central moment of the second order.

As starting-point for the determination of the correlation distribution of

$$z_1 = \frac{1}{N} [x_1^2 + x_2^2 + x_3^2 + \dots + x_N^2]$$

and

$$z_2 = \frac{1}{N} [x_1 + x_2 + x_3 + \dots + x_N],$$

it is most suitable to take the characteristic function. The characteristic function $U(t_1, t_2)$ of the distribution $F(z_1, z_2)$ of z_1 and z_2 is given by the formula

$$U(t_1, t_2) = \left[\int_{-\infty}^{\infty} F(x) e^{\frac{x^2}{N} t_1 + \frac{x}{N} t_2} dx \right]^N.$$

We have, therefore, to evaluate the integral

$$\int_{-\infty}^{\infty} F(x) e^{\frac{x^2}{N} t_1 + \frac{x}{N} t_2} dx = \int_{-\infty}^{\infty} \varphi(x) e^{\frac{x^2}{N} t_1 + \frac{x}{N} t_2} dx + \sum \frac{A_n}{n!} \int_{-\infty}^{\infty} H_n(x) \varphi(x) e^{\frac{x^2}{N} t_1 + \frac{x}{N} t_2} dx.$$

The first integral will be, as before, (formula 33.)

$$(45) \quad \left[1 - \frac{2\lambda_2}{N} t_1 \right]^{-\frac{1}{2}} e^{\frac{\lambda_2 t_2^2}{2N^2 \left(1 - \frac{2\lambda_2}{N} t_1 \right)}} = U_0(t_1, t_2),$$

while the integral

$$I_n = \int_{-\infty}^{\infty} H_n(x) \varphi(x) e^{\frac{x^2}{N} t_1 + \frac{x}{N} t_2} dx$$

after linear transformation becomes

$$I_n = U_0(t_1, t_2) \int_{-\infty}^{\infty} H_n \left(x - \frac{1}{N} \frac{\lambda_2 t_2}{1 - \frac{2\lambda_2 t_1}{N}} \right) \cdot \frac{1}{\sqrt{\frac{2\pi\lambda_2}{\left(1 - \frac{2\lambda_2 t_1}{N}\right)}}} e^{-\frac{1 - \frac{2\lambda_2}{N} t_1}{2\lambda_2} x^2} dx.$$

Using the formula (17) in *Chapter 3*

$$I_n = U_0(t_1, t_2) \left[\sum_{i=0}^n \binom{n}{i} \left(\frac{t_2}{N \left(1 - \frac{2\lambda_2}{N} t_1\right)} \right)^i \int_{-\infty}^{\infty} H_{n-i}(x) e^{-\frac{1 - \frac{2\lambda_2}{N} t_1}{2\lambda_2} x^2} dx \right]$$

and applying the formula (19) in *Chapter 3*, we obtain

$$(46 \text{ a}) \quad \frac{I_{2n}}{2n!} = U_0(t_1, t_2) \left[\frac{1}{n!} \left(\frac{\tau_1}{N} \right)^n + \frac{1}{(n-1)! 2!} \left(\frac{\tau_1}{N} \right)^{n-1} \left(\frac{\tau_2}{N} \right)^2 + \dots + \frac{1}{2n!} \left(\frac{\tau_2}{N} \right)^{2n} \right]$$

$$(46 \text{ b}) \quad \frac{I_{2n+1}}{(2n+1)!} = U_0(t_1, t_2) \left[\frac{1}{n! 1!} \left(\frac{\tau_1}{N} \right)^n \left(\frac{\tau_2}{N} \right) + \right. \\ \left. + \frac{1}{(n-1)! 3!} \left(\frac{\tau_1}{N} \right)^{n-1} \left(\frac{\tau_2}{N} \right)^3 + \dots + \frac{1}{(2n+1)!} \left(\frac{\tau_2}{N} \right)^{2n+1} \right],$$

where

$$\tau_1 = \frac{t_1}{1 - \frac{2\lambda_2}{N} t_1}, \quad \tau_2 = \frac{t_2}{1 - \frac{2\lambda_2}{N} t_1}.$$

Hence we obtain

$$(47 \text{ a}) \quad U(t_1, t_2) = \left[1 - \frac{2\lambda_2}{N} t_1 \right]^{-\frac{N}{2} \frac{\lambda_2}{2N} \frac{t_2^2}{1 - \frac{2\lambda_2}{N} t_1}} \cdot \left[1 + \sum \sum \frac{A_{2i+j}}{N^{i+j} i! j!} \tau_1^i \tau_2^j \right]^N$$

or

$$(47 \text{ b}) \quad U(t_1, t_2) = \left[1 - \frac{2\lambda_2}{N} t_1 \right]^{-\frac{N}{2} \frac{\lambda_2}{2N} \frac{t_2^2}{1 - \frac{2\lambda_2}{N} t_1}} \cdot \left[1 + \sum \sum \frac{B_{ij}}{i! j!} \tau_1^i \tau_2^j \right].$$

From the characteristic function the distribution of $z_1 = v_2'$ and $z_2 = v_1'$ will then be obtained by the »Inversion Theorem».

$$F(z_1, z_2) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(\omega_1 i, \omega_2 i) e^{-z_1 \omega_1 i - z_2 \omega_2 i} d\omega_1 d\omega_2.$$

If this expression is integrated term by term, we have to evaluate expressions of the type

$$(48) \quad \frac{B_{ij}}{i!j!} \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[1 - \frac{2\lambda_2}{N} \omega_1 i\right]^{-\frac{N}{2} \frac{\lambda_2}{2N} \frac{\overline{\omega_2 i}^2}{1 - \frac{2\lambda_2}{N} \omega_1 i}} e^{-z_1 \omega_1 i - z_2 \omega_2 i} d\omega_1 d\omega_2.$$

If this expression is integrated, first with regard to the variate ω_2 , we know that

$$(49) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{\lambda_2 \overline{\omega_2 i}^2}{2N(1 - \frac{2\lambda_2}{N} \omega_1 i)}} \omega_2 i e^{-z_2 \omega_2 i} d\omega_2$$

is the j th derivative of a normal distribution in which the second seminvariant is

$$\frac{\lambda_2}{N \left[1 - \frac{2\lambda_2}{N} \omega_1 i\right]},$$

or the product of a normal distribution with a Hermitian polynomial.

The normal distribution is

$$\sqrt{\frac{N \left(1 - \frac{2\lambda_2 \omega_1 i}{N}\right)}{2\pi\lambda_2}} e^{-\frac{N \left(1 - \frac{2\lambda_2 \omega_1 i}{N}\right) z_2^2}{2\lambda_2}} = \frac{\sqrt{N}}{\sqrt{2\pi\lambda_2}} e^{-\frac{N}{2\lambda_2} z_2^2} \cdot \left(1 - \frac{2\lambda_2 \omega_1 i}{N}\right)^{\frac{1}{2}} e^{\omega_1 i z_2^2}.$$

But as the corresponding Hermitian Polynomials with the parameter

$$\frac{\lambda_2}{N \left[1 - \frac{2\lambda_2 \omega_1 i}{N}\right]} (= \lambda_2')$$

can according to formula (18) in *Chapter 3* be transformed into a series of other Hermitian polynomials with the parameter $\frac{\lambda_2}{N}$, we obtain

$$\begin{aligned} \lambda_2^j H_j(z_2) = & \left[1 - \frac{2\lambda_2}{N} \omega_1 i \right]^j \left[\lambda_2 H_j(z_2) + \frac{j(j-1)}{1} \frac{\omega_1 i}{1 - \frac{2\lambda_2}{N} \omega_1 i} \lambda_2 H_{j-2}(z_2) + \right. \\ & \left. + \frac{j(j-1)(j-2)(j-3)}{2!} \left(\frac{\omega_1 i}{1 - \frac{2\lambda_2}{N} \omega_1 i} \right)^2 \lambda_2 H_{j-4}(z_2) + \dots \right]. \end{aligned}$$

Thus the integral (49) can be written as an expression containing derivatives of j th and lower orders of

$$\varphi(z_2) = \sqrt{\frac{N}{2\pi\lambda_2}} e^{-\frac{N}{2\lambda_2} z_2^2}$$

$$\begin{aligned} & \left[1 - \frac{2\lambda_2}{N} \omega_1 i \right] e^{z_2^2 \omega_1 i} \left[(-1)^j \frac{1}{j!} \varphi^{(j)}(z_2) + \right. \\ & + (-1)^{j-2} \frac{1}{1!(j-2)!} \varphi^{(j-2)}(z_2) \left(\frac{\omega_1 i}{1 - \frac{2\lambda_2}{N} \omega_1 i} \right) + \\ & \left. + (-1)^{j-4} \frac{1}{2!(j-4)!} \varphi^{(j-4)}(z_2) \left(\frac{\omega_1 i}{1 - \frac{2\lambda_2}{N} \omega_1 i} \right)^2 + \dots \right]. \end{aligned}$$

The double integral (48) can finally be written as an expression of the form

$$\begin{aligned} & B_{ij} \left[(-1)^{i+j} \frac{1}{i!j!} f_{\frac{N-1}{2}+i}^{(i)}(z_1 - z_2^2) \varphi^{(j)}(z_2) + \right. \\ & \left. + (-1)^{i+j-1} \frac{1}{(i+1)!(j-2)!} \binom{i+1}{1} f_{\frac{N-1}{2}+i+1}^{(i+1)}(z_1 - z_2^2) \varphi^{(j-2)}(z_2) + \dots \right], \end{aligned}$$

and after a certain regrouping we find that the distribution of the first moment and of the central moment of the second order, $\nu_2 = z_1 - z_2^2$, becomes a combined distribution of »Type A» and »Type III».

$$F(\nu_1', \nu_2) = f_{\frac{N-1}{2}}(\nu_2) \varphi(\nu_1') + \sum \sum (-1)^{i+j} \frac{C_{ij}}{i!j!} f_{\frac{N-1}{2}+i}^{(i)}(\nu_2) \varphi^{(j)}(\nu_1'),$$

where

$$(51) \quad C_{ij} = B_{ij} - \binom{i}{1} B_{i-1, j+2} + \binom{i}{2} B_{i-2, j+4} - \dots + (-1)^i B_{0, j+2i}.$$

The distribution of the central moment of the second order becomes

$$(52) \quad F(v_2) = f_{\frac{N-1}{2}}(v_2) + \sum (-1)^i \frac{C_{i0}}{i!} f_{\frac{N-1}{2}+i}^{(i)}(v_2).$$

The main term of these distributions is the one that is obtained from the main term in the given *A-Series*, and we observe that this term and accordingly the different derivatives depend only on the second order seminvariant of the parent population. Conversely, the coefficients belonging to the different derivatives depend only on the *A*-coefficients of the given frequency distribution and thus only on the seminvariants of the third or higher order.

We thus find that when the distribution of the parent population is a series of »Type A» the distribution of the second moment is a »Romanowsky's Generalization of Type III» and that the correlation distribution of the first and second moment is a combination between »Gram-Charlier's A-Series» and »Romanowsky's Generalization of Type III».

CHAPTER 6.

The Characteristics of the Correlation Distribution of the First and Second Moments.

For the coefficients C_{ij} defined in *Chapter 5* (51) we find the following expressions up to the sixth order.

$$\begin{aligned}
 C_{10} &= 0 \\
 C_{01} &= 0 \\
 &\dots\dots\dots \\
 C_{20} &= \frac{(N-1)^2}{N^3} \cdot \lambda_4 \\
 C_{11} &= \frac{N-1}{N^2} \cdot \lambda_3 \\
 C_{02} &= 0 \\
 &\dots\dots\dots \\
 C_{30} &= \frac{(N-1)^3}{N^5} \lambda_6 + 4 \frac{(N-1)(N-2)}{N^4} \lambda_3^2 \\
 (53) \quad C_{21} &= \frac{(N-1)^2}{N^4} \lambda_5 \\
 C_{12} &= \frac{(N-1)}{N^3} \lambda_4 \\
 C_{03} &= \frac{1}{N^2} \lambda_3 \\
 &\dots\dots\dots \\
 C_{40} &= \frac{(N-1)^4}{N^7} \lambda_8 + 32 \frac{(N-1)^2(N-2)}{N^6} \lambda_5 \lambda_3 + \\
 &\quad + \frac{(N-1)(3N^3 + 23N^2 - 63N + 45)}{N^6} \lambda_4^2 \\
 C_{31} &= \frac{(N-1)^3}{N^6} \lambda_7 + \frac{(N-1)(3N^2 + 14N - 25)}{N^5} \lambda_4 \lambda_3
 \end{aligned}$$

$$C_{22} = \frac{(N-1)^2}{N^5} \lambda_6 + 2 \frac{(N-1)(N+1)}{N^4} \lambda_3^2$$

$$C_{13} = \frac{(N-1)}{N^4} \lambda_5$$

$$C_{04} = \frac{1}{N^3} \lambda_4$$

.....

$$C_{50} = \frac{(N-1)^5}{N^9} \lambda_{10} + 80 \frac{(N-1)^3(N-2)}{N^8} \lambda_7 \lambda_3 +$$

$$+ 10 \frac{(N-1)^2(N^3 + 17N^2 - 45N + 35)}{N^8} \lambda_6 \lambda_4 + 16 \frac{(N-1)(N-2)(6N^2 - 12N + 7)}{N^8} \lambda_5^2 +$$

$$+ 40 \frac{(N-1)(N-2)(N^2 + 22N - 35)}{N^7} \lambda_4 \lambda_3^2$$

$$C_{41} = \frac{(N-1)^4}{N^8} \lambda_9 + 4 \frac{(N-1)^2(N^2 + 12N - 21)}{N^7} \lambda_6 \lambda_3 +$$

$$+ \frac{(N-1)(6N^3 + 78N^2 - 222N + 154)}{N^7} \lambda_5 \lambda_4 + 16 \frac{(N-1)(N^2 + 3N - 10)}{N^6} \lambda_3^3$$

$$C_{32} = \frac{(N-1)^3}{N^7} \lambda_8 + \frac{(N-1)(6N^2 + 20N - 34)}{N^6} \lambda_5 \lambda_3 +$$

$$(53) \quad + \frac{(N-1)(3N^2 + 14N - 25)}{N^6} \lambda_4^2$$

$$C_{23} = \frac{(N-1)^2}{N^6} \lambda_7 + \frac{(N-1)(7N + 5)}{N^5} \lambda_4 \lambda_3$$

$$C_{14} = \frac{(N-1)}{N^5} \lambda_6 + 4 \frac{(N-1)}{N^4} \lambda_3^2$$

$$C_{05} = \frac{1}{N^4} \lambda_5$$

.....

$$C_{60} = \frac{(N-1)^6}{N^{11}} \lambda_{12} + 160 \frac{(N-1)^4(N-2)}{N^{10}} \lambda_9 \lambda_3 +$$

$$+ 15 \frac{(N-1)^3(N^3 + 29N^2 - 77N + 63)}{N^{10}} \lambda_8 \lambda_4 +$$

$$+ 96 \frac{(N-1)^2(7N^3 - 28N^2 + 37N - 18)}{N^{10}} \lambda_7 \lambda_5 +$$

$$+ \frac{(N-1)(10N^5 + 402N^4 - 1980N^3 + 3700N^2 - 3150N + 1050)}{N^{10}} \lambda_6^2 +$$

$$+ 80 \frac{(N-1)^2(N-2)(N^2 + 60N - 105)}{N^9} \lambda_6 \lambda_3^2 +$$

$$\begin{aligned}
& + 480 \frac{(N-1)(N-2)(N^3 + 33N^2 - 89N + 63)}{N^9} \lambda_5 \lambda_4 \lambda_3 + \\
& + 15 \frac{(N-1)(N^5 + 27N^4 + 226N^3 - 1098N^2 + 1725N - 945)}{N^9} \lambda_4^3 + \\
& + 160 \frac{(N-1)(N-2)(N^2 + 27N - 70)}{N^8} \lambda_3^4 \\
C_{51} = & \frac{(N-1)^5}{N^{10}} \lambda_{11} + 5 \frac{(N-1)^3(N^2 + 22N - 39)}{N^9} \lambda_8 \lambda_3 + \\
& + 10 \frac{(N-1)^2(N^3 + 25N^2 - 69N + 51)}{N^9} \lambda_7 \lambda_4 + \\
& + \frac{(N-1)(10N^4 + 352N^3 - 1388N^2 + 1792N - 798)}{N^9} \lambda_6 \lambda_5 + \\
& + 20 \frac{(N-1)(N-2)(10N^2 + 92N - 126)}{N^8} \lambda_5 \lambda_3^2 + \\
& + \frac{(N-1)(15N^4 + 300N^3 + 2090N^2 - 8300N + 7175)}{N^8} \lambda_4^2 \lambda_3 \\
(53) \quad C_{42} = & \frac{(N-1)^4}{N^9} \lambda_{10} + 8 \frac{(N-1)^2(N^2 + 8N - 13)}{N^8} \lambda_7 \lambda_3 + \\
& + \frac{(N-1)(10N^3 + 122N^2 - 354N + 238)}{N^9} \lambda_6 \lambda_4 + \\
& + \frac{(N-1)(6N^3 + 78N^2 - 222N + 154)}{N^8} \lambda_5^2 + \\
& + \frac{(N-1)(12N^3 + 164N^2 + 324N - 980)}{N^7} \lambda_4 \lambda_3^2 \\
C_{33} = & \frac{(N-1)^3}{N^8} \lambda_9 + \frac{(N-1)(10N^2 + 24N - 42)}{N^7} \lambda_6 \lambda_3 + \\
& + 12 \frac{(N-1)(N^2 + 4N - 7)}{N^7} \lambda_5 \lambda_4 + \frac{(N-1)(6N^2 + 28N + 10)}{N^6} \lambda_3^3 \\
C_{24} = & \frac{(N-1)^2}{N^7} \lambda_8 + \frac{(N-1)(12N + 4)}{N^6} \lambda_5 \lambda_3 + \frac{(N-1)(7N + 5)}{N^6} \lambda_4^2 \\
C_{15} = & \frac{(N-1)}{N^6} \lambda_7 + 15 \frac{(N-1)}{N^5} \lambda_4 \lambda_3 \\
C_{06} = & \frac{1}{N^5} \lambda_6 + 10 \frac{1}{N^4} \lambda_3^2
\end{aligned}$$

From these C_{ij} coefficients we define the following characteristics

$$(54) \quad e^{\sum \sum \frac{\vartheta_{ij}}{i!j!} \tau_1^i \tau_2^j} = 1 + \sum \sum \frac{C_{ij}}{i!j!} \tau_1^i \tau_2^j,$$

which are a combination of the general seminvariants and the »Type III cumulants» previously introduced. These characteristics will undoubtedly have the same property of mutual independence in samples from an uncorrelated two-dimensional distribution of the combined normal and »Type III» distribution as the seminvariants and the »Type III cumulants» will have in samples from a one-dimensional distribution of normal type or »Type III».

We can pass from these to the general seminvariants by the formula

$$(55) \quad \sum \sum \frac{\vartheta_{ij}}{i!j!} \left(\frac{t_1}{1 - \frac{2\lambda_2}{N} t_1} \right)^i t_2^j = \sum \sum \frac{L_{ij}}{i!j!} t_1^i t_2^j.$$

From the above it is evident that the expressions for ϑ_{ij} include the seminvariants of the original distribution of order $2i + j$ and lower, with the exception of the two lowest, λ_1 and λ_2 , while λ_2 also forms part of the expressions for the seminvariants of the distribution, which will consequently be more complicated.

It should be observed that from these relations it follows immediately that all the terms of the formula for L_{ij} , which involves a power of λ_2 , can be derived from the formulas for the seminvariants of lower order. (This fact has been observed earlier, but as far as I know no satisfactory proof has been given.¹)

We obtain the following values of the ϑ_{ij} characteristics in terms of the original seminvariants up to the sixth order.

$$(56) \quad \begin{aligned} \vartheta_{10} &= 0 \\ \vartheta_{01} &= 0 \\ &\dots\dots\dots \\ \vartheta_{20} &= \frac{(N-1)^2}{N^3} \lambda_4 \\ \vartheta_{11} &= \frac{(N-1)}{N^2} \lambda_3 \\ \vartheta_{02} &= 0 \\ &\dots\dots\dots \end{aligned}$$

¹ J. WISHART. A Comparison of the Semi-invariants or the Distributions of Moment and Semi-invariant Estimates in Samples from an Infinite Population. *Biometrika*. 25. 1933.

$$\begin{aligned}
\vartheta_{30} &= \frac{(N-1)^3}{N^5} \lambda_6 + 4 \frac{(N-1)(N-2)}{N^4} \lambda_3^2 \\
\vartheta_{21} &= \frac{(N-1)^2}{N^4} \lambda_5 \\
\vartheta_{12} &= \frac{(N-1)}{N^3} \lambda_4 \\
\vartheta_{03} &= \frac{1}{N^2} \lambda_3 \\
&\dots\dots\dots \\
\vartheta_{40} &= \frac{(N-1)^4}{N^7} \lambda_8 + 32 \frac{(N-1)^2(N-2)}{N^6} \lambda_5 \lambda_3 + 8 \frac{(N-1)(4N^2-9N+6)}{N^6} \lambda_4^2 \\
\vartheta_{31} &= \frac{(N-1)^3}{N^6} \lambda_7 + 4 \frac{(N-1)(5N-7)}{N^5} \lambda_4 \lambda_3 \\
\vartheta_{22} &= \frac{(N-1)^2}{N^5} \lambda_6 + 4 \frac{(N-1)}{N^4} \lambda_3^2 \\
\vartheta_{13} &= \frac{(N-1)}{N^4} \lambda_5 \\
\vartheta_{04} &= \frac{1}{N^3} \lambda_4 \\
(56) \quad &\dots\dots\dots \\
\vartheta_{50} &= \frac{(N-1)^5}{N^9} \lambda_{10} + 80 \frac{(N-1)^3(N-2)}{N^8} \lambda_7 \lambda_3 + \\
&\quad + 40 \frac{(N-1)^2(5N^2-12N+9)}{N^8} \lambda_6 \lambda_4 + \\
&\quad + 16 \frac{(N-1)(N-2)(6N^2-12N+7)}{N^8} \lambda_5^2 + \\
&\quad + 480 \frac{(N-1)(N-2)(2N-3)}{N^7} \lambda_4 \lambda_3^2 \\
\vartheta_{41} &= \frac{(N-1)^4}{N^8} \lambda_9 + 8 \frac{(N-1)^2(7N-11)}{N^7} \lambda_6 \lambda_3 + \\
&\quad + 16 \frac{(N-1)(6N^2-15N+10)}{N^7} \lambda_5 \lambda_4 + 96 \frac{(N-1)(N-2)}{N^6} \lambda_3^3 \\
\vartheta_{32} &= \frac{(N-1)^3}{N^7} \lambda_8 + 8 \frac{(N-1)(4N-5)}{N^6} \lambda_5 \lambda_3 + 4 \frac{(N-1)(5N-7)}{N^6} \lambda_4^2 \\
\vartheta_{23} &= \frac{(N-1)^2}{N^6} \lambda_7 + 12 \frac{(N-1)}{N^5} \lambda_4 \lambda_3 \\
\vartheta_{14} &= \frac{(N-1)}{N^5} \lambda_6
\end{aligned}$$

$$\vartheta_{05} = \frac{1}{N^4} \lambda_5$$

.....

$$\begin{aligned} \vartheta_{60} = & \frac{(N-1)^6}{N^{11}} \lambda_{12} + 160 \frac{(N-1)^4 (N-2)}{N^{10}} \lambda_9 \lambda_3 + \\ & + 240 \frac{(N-1)^3 (2N^2 - 5N + 4)}{N^{10}} \lambda_8 \lambda_4 + \\ & + 96 \frac{(N-1)^2 (N-2) (7N^2 - 14N + 9)}{N^{10}} \lambda_7 \lambda_5 + \\ & + 4 \frac{(N-1) (113N^4 - 520N^3 + 950N^2 - 800N + 265)}{N^{10}} \lambda_6^2 + \\ & + 160 \frac{(N-1)^2 (N-2) (31N - 53)}{N^9} \lambda_6 \lambda_3^2 + \\ & + 1920 \frac{(N-1) (N-2) (9N^2 - 23N + 16)}{N^9} \lambda_5 \lambda_4 \lambda_3 + \\ & + 480 \frac{(N-1) (11N^3 - 41N^2 + 59N - 31)}{N^9} \lambda_4^3 + \\ (56) \quad & + 960 \frac{(N-1) (N-2) (5N - 12)}{N^8} \lambda_3^4 \end{aligned}$$

$$\begin{aligned} \vartheta_{51} = & \frac{(N-1)^5}{N^{10}} \lambda_{11} + 20 \frac{(N-1)^3 (6N - 10)}{N^9} \lambda_8 \lambda_3 + \\ & + 40 \frac{(N-1)^2 (7N^2 - 18N + 13)}{N^9} \lambda_7 \lambda_4 + \\ & + 4 \frac{(N-1) (98N^3 - 362N^2 + 458N - 202)}{N^9} \lambda_6 \lambda_5 + \\ & + 160 \frac{(N-1) (N-2) (14N - 17)}{N^8} \lambda_5 \lambda_3^2 + \\ & + 640 \frac{(N-1) (5N^2 - 15N + 12)}{N^8} \lambda_4^2 \lambda_3 \end{aligned}$$

$$\begin{aligned} \vartheta_{42} = & \frac{(N-1)^4}{N^9} \lambda_{10} + 16 \frac{(N-1)^2 (5N - 7)}{N^8} \lambda_7 \lambda_3 + \\ & + 8 \frac{(N-1) (19N^2 - 48N + 31)}{N^8} \lambda_6 \lambda_4 + 16 \frac{(N-1) (6N^2 - 15N + 10)}{N^8} \lambda_5^2 + \\ & + 96 \frac{(N-1) (8N - 13)}{N^8} \lambda_4 \lambda_3^2 \end{aligned}$$

$$\begin{aligned}
 \vartheta_{33} &= \frac{(N-1)^3}{N^8} \lambda_9 + 4 \frac{(N-1)(11N-13)}{N^7} \lambda_6 \lambda_3 + \\
 &\quad + 24 \frac{(N-1)(3N-4)}{N^7} \lambda_5 \lambda_4 + \frac{48(N-1)}{N^6} \lambda_3^3 \\
 (56) \quad \vartheta_{24} &= \frac{(N-1)^2}{N^7} \lambda_8 + 16 \frac{(N-1)}{N^6} \lambda_5 \lambda_3 + 12 \frac{(N-1)}{N^6} \lambda_4^2 \\
 \vartheta_{15} &= \frac{(N-1)}{N^6} \lambda_7 \\
 \vartheta_{06} &= \frac{1}{N^5} \lambda_6.
 \end{aligned}$$

I am not giving the formulas for the corresponding seminvariants. Certainly they can be obtained without difficulty from the given relations between them and these ϑ_{ij} characteristics, but, as it is ultimately these ϑ_{ij} characteristics which determine the connection between the mean and the second central moment, the seminvariants are here of subordinate importance.

The formulas for the given ϑ_{ij} characteristics indicate that ϑ_{1n} and consequently also L_{1n} are given by the formula $\frac{N-1}{N^{n+1}} \lambda_{n+2}$ (which has been demonstrated before, e. g. by R. A. Fisher¹⁾) and that simple relations also exist for the higher coefficients.

The coefficients C_{ij} and accordingly ϑ_{ij} can also be derived in a more general way. The coefficients B_{ij} have been defined by the formula

$$(57) \quad \sum \sum \frac{B_{ij}}{i!j!} \tau_1^i \tau_2^j = \left[1 + \sum \frac{A_n}{N^n n!} t_2^n + \frac{\tau_1}{N1!} \sum \frac{A_{n+2}}{N^n n!} t_2^n + \dots \right]^N.$$

If the expression $1 + \sum \frac{A_n}{N^n n!} t_2^n$ is denoted by $u(t_2) = e^{f(t_2)}$, where

$$f(t_2) = \sum_n \frac{\lambda_n}{N^n n!} t_2^n,$$

we obtain

$$\sum \frac{A_{n+2i}}{N^n n!} t_2^n = N^{2i} \frac{\delta^{2i} u(t_2)}{\delta t_2^{2i}}$$

and the expression for the B_{ij} coefficients will be

$$(58) \quad \sum \sum \frac{B_{ij}}{i!j!} \tau_1^i \tau_2^j = \left[u(t_2) + \frac{N\tau_1}{1!} \frac{\delta^2 u}{\delta t_2^2} + \frac{N^2 \tau_1^2}{2!} \cdot \frac{\delta^4 u}{\delta t_2^4} + \dots \right]^N = [U(\tau_1, t_2)]^N.$$

¹ R. A. FISHER: »Moments and Product Moments». Proceedings of the London Mathematical Society, Ser. 2, XXX, 1928.

As

$$C_{ij} = B_{ij} - \binom{i}{1} B_{i-1, j+2} + \dots (-1)^i B_{0, j+2i},$$

we can write

$$\begin{aligned} \sum \sum \frac{C_{ij}}{i! j!} \tau_1^i \tau_2^j &= \sum \sum \frac{B_{ij}}{i! j!} \tau_1^i \tau_2^j - \tau_1 \cdot \frac{\delta^2}{\delta t_2^2} \sum \sum \frac{B_{ij}}{i! j!} \tau_1^i \tau_2^j + \\ &+ \frac{\tau_1^2}{2!} \cdot \frac{\delta^4}{\delta t_2^4} \sum \sum \frac{B_{ij}}{i! j!} \tau_1^i \tau_2^j - \dots \end{aligned}$$

or

$$(59) \quad \sum \sum \frac{C_{ij}}{i! j!} \tau_1^i \tau_2^j = [U(\tau_1, \tau_2)]^N - \tau_1 \cdot \frac{\delta^2}{\delta t_2^2} U^N + \frac{\tau_1^2}{2!} \cdot \frac{\delta^4}{\delta t_2^4} U^N - \dots$$

These relations ultimately give us the following expressions for the coefficients C_{ij}

$$\begin{aligned} \sum \frac{C_{0n}}{n!} t^n &= e^{Nf(t)} \\ (60) \quad \sum \frac{C_{1n}}{n!} t^n &= e^{Nf(t)} N(N-1) f^{(2)}(t) \\ \sum \frac{C_{2n}}{n!} t^n &= e^{Nf(t)} [N(N-1)^2 f^{(4)}(t) + N^2(N-1)(N+1)(f^{(2)}(t))^2] \end{aligned}$$

From these formulas we obtain

$$\begin{aligned} \sum \frac{\vartheta_{0n}}{n!} t^n &= Nf(t) \\ \sum \frac{\vartheta_{1n}}{n!} t^n &= N(N-1)f^{(2)}(t) \\ \sum \frac{\vartheta_{2n}}{n!} t^n &= N(N-1)^2 f^{(4)}(t) + 2N^2(N-1)[f^{(2)}(t)]^2 \\ (61) \quad \sum \frac{\vartheta_{3n}}{n!} t^n &= N(N-1)^3 f^{(6)}(t) + 12N^2(N-1)^2 f^{(4)}(t) f^{(2)}(t) + \\ &+ 4N^2(N-1)(N-2)[f^{(3)}(t)]^2 + 8N^3(N-1)[f^{(2)}(t)]^3 \\ \sum \frac{\vartheta_{4n}}{n!} t^n &= N(N-1)^4 f^{(8)}(t) + 24N^2(N-1)^3 f^{(6)}(t) f^{(2)}(t) + \\ &+ 32N^2(N-1)^2(N-2)f^{(5)}(t)f^{(3)}(t) + 8N^2(N-1)[4N^2 - 9N + 6][f^{(4)}(t)]^2 + \\ &+ 144N^3(N-1)^2 f^{(4)}(t)[f^{(2)}(t)]^2 + 96N^3(N-1)(N-2)[f^{(3)}(t)]^2 f^{(2)}(t) + \\ &+ 180N^4(N-1)[f^{(2)}(t)]^4. \end{aligned}$$

Or

$$\begin{aligned}
 \vartheta_{0n} &= \frac{\lambda_n}{N^{n-1}} \\
 (62) \quad \vartheta_{1n} &= \frac{N-1}{N^{n+1}} \lambda_{n+2} \\
 \vartheta_{2n} &= \frac{(N-1)^2}{N^{n+3}} \lambda_{n+4} + \frac{2(N-1)}{N^{n+2}} \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} \lambda_{i+2} \lambda_{n-i+2}.
 \end{aligned}$$

It will be noted that the first derivative of $f(t)$ always vanishes in these expressions (61). Although I refrain from giving proof that this is also the case for the coefficients of higher degree, I venture to assert that this must be so on the following account. In the deduction given above no supposition whatever was made regarding the first two coefficients in the A -Series. There is no necessity to choose the two parameters a and b in the formula for the normal curve equal to the two first seminvariants, it being possible to assign other values to the parameters, although in this case the two first terms of the A -Series do not vanish and in the expression

$$f(t_2) = \sum \frac{\lambda^n}{n! N^n} t_2^n$$

the two first terms do not both vanish.

The second moment about the mean of the sample, however, is independent of whatever starting-point has been chosen, and therefore the coefficients cannot include the first seminvariant. Hence there cannot be in the expression for ϑ_{ij} any term containing the first derivative $f^{(1)}(t_2)$, which is the only derivative that depends on the situation of the zero-point.

CHAPTER 7.

The Magnitude of the different Terms of the Frequency Distribution of the Second Moment.

As previously stated, provided certain conditions are fulfilled, an infinite »Gram-Charliers *A-Series*» is convergent for all values of x .

The series that describes the frequency distribution for the square of the variate, $z = x^2$, must accordingly also be convergent for all values of z (except $z = 0$ where $F(z) = \infty$).

As Romanowsky states that a frequency function, defined in the interval $0 - \infty$, may always be developed into a convergent infinite series of his type, the convergency of the infinite series will not be discussed.

The following short reference may here be made to the magnitude of the different terms if only a finite number of terms are included, and more especially to how the terms behave when the number of individuals in the random sample becomes larger.

The coefficients B_{n0} in the distribution of the second moment about zero are of the following magnitude in relation to N :

$$B_{20} \sim \frac{1}{N}$$

$$B_{30}, B_{40} \sim \frac{1}{N^2}$$

$$B_{50}, B_{60} \sim \frac{1}{N^3},$$

and we obtain the general relation

$$B_{2n, 0} \sim \frac{1}{N^n}$$

$$B_{2n+1, 0} \sim \frac{1}{N^{n+1}}.$$

The coefficients C_{n0} in the distribution of the second central moment assume the same order as B_{n0} , since the coefficients $B_{n-i, 2i}$ are of the same order or lower in comparison with B_{n0} .

Hence

$$C_{2n,0} \sim \frac{1}{N^n}$$

$$C_{2n+1,0} \sim \frac{1}{N^{n+1}}.$$

The different terms $(-1)^n f_{\frac{N-1}{2}+n}^{(n)}(z)$ increase however as N increases, since the polynomials

$$L_n(z) = \left(\frac{N}{2}\right)^n \left[\frac{z^n N^n}{(N-1)(N+1)\dots(N+2n-3)} - \right. \\ \left. - \binom{n}{1} \frac{z^{n-1} N^{n-1}}{(N-1)\dots(N+2n-5)} + \dots + (-1)^n \right]$$

increase with N . (For the sake of simplicity the second seminvariant λ_2 has been taken here as unity.)

If only the values of z which deviate from the value $\frac{N-1}{N}$ (the mean value of z) by the order of magnitude $\frac{1}{\sqrt{N}}$ are considered, these polynomials are of the order $N^{n/2}$, which can readily be proved in the following way.

Put

$$P_i(z) = N^i z^i - \binom{i}{1} (N+2n-3) N^{i-1} z^{i-1} + \binom{i}{2} (N+2n-3)(N+2n-5) N^{i-2} z^{i-2} - \dots + (-1)^i (N+2n-3)(N+2n-5)\dots(N+2n-2i-1).$$

If

$$z = \frac{N-1}{N} + u; \text{ where } u \sim \frac{1}{\sqrt{N}}$$

we get

$$P_i(z) = [uN - 2(n-i)] P_{i-1}(z) - 2(i-1)Nz P_{i-2}(z).$$

As

$$P_0(z) \sim N^0 \text{ and } P_1(z) \sim N^{\frac{1}{2}} \text{ we get } P_i(z) \sim N^{\frac{i}{2}},$$

and in consequence the polynomial $L_n(z)$ assumes the order $N^{n/2}$. Hence the coefficients decrease generally with $N^{\frac{n}{2}}$ and the derivatives increase with $N^{n/2}$; therefore, the different terms in the frequency function do not decrease as N increases.

The even terms will assume the order N^0 and the odd terms the order $N^{-1/2}$.

Should, however, the parent population be of such a nature that the coefficient $A_4 = 0$, that is, $\lambda_4 = 0$, the coefficients B_{20} would vanish and in that case B_{i0} would be of the order $N^{-2i/3}$, if the exponent is raised to the immediately higher power. In such a case the successive terms will decrease in value as N increases.

It follows that in most cases the error distribution with a growing N does not tend to the »Pearson's Type III» function that describes the distribution of the second moment in a random sample from a normal parent population.

This, of course, is obvious, since the second moment of the distribution has the appearance $\frac{N-1}{N^2} [\lambda_4 + 2\lambda_2^2]$, which expression does not tend to $\frac{N-1}{N^2} 2\lambda_2^2$ with a rising N .

If it is postulated that the parent frequency distribution has been generated under the influence of a number (s) of elementary errors, then the seminvariants λ_n (or, more correctly, the standardized seminvariants γ_n) will have the order $s^{-\left[\frac{n}{2}-1\right]}$ and the coefficients A_n will assume the following orders of magnitude

$$A_{3i-4}, A_{3i-2}, A_{3i} \sim \frac{1}{s^{i/2}}.$$

The coefficients B_{n0} and with them also the coefficients C_{n0} will attain the following orders

$$B_{20}, B_{30} \sim \frac{1}{s}$$

$$B_{40}, B_{50}, B_{60} \sim \frac{1}{s^2}$$

and generally

$$B_{3n-2,0}, B_{3n-1,0}, B_{3n,0} \sim \frac{1}{s^n}.$$

Thus, broadly viewed, the coefficients will decrease doubly as rapidly as the coefficients in the A -Series. *Consequently, it is clear that the frequency distribution found for the second moment can be considered to have a significance only for the case of a parent population that can be practicably described by a »Gram-Charlier's A -Series» in which the coefficients diminish rather rapidly; in other words, the deviations from the normal frequency curve must not be too great.*

CHAPTER 8.

The Distribution of the Ratio between the Deviation of the Sample Mean from the Mean of the Parent Population and the Standard Deviation in the Sample.

The distribution of the difference between the means of the sample and the parent population, ν_1' , and the squared standard deviation in the sample, ν_2 , is in accordance with the foregoing given in the form of a series. We shall here consider only the distribution of the absolute value of ν_1' , and the distribution of $|\nu_1'|$ and ν_2 will be

$$F(\nu_2, |\nu_1'|) = 2 \left[\frac{f_{N-1}(\nu_2)}{2} \varphi(|\nu_1'|) + \sum \sum (-1)^i \frac{C_{i, 2j}}{i! 2j!} f_{\frac{N-1}{2}+i}^{(i)}(\nu_2) \varphi^{(2j)}(|\nu_1'|) \right].$$

The distribution $F(t)$ of the relation $t = \frac{|\nu_1'|}{\sqrt{\nu_2}}$ is obtained by the formula

$$F(t) = \int_0^\infty F(\nu_2, t\sqrt{\nu_2}) \sqrt{\nu_2} d\nu_2.$$

The integral $\int_0^t F(t) dt$ then indicates the probability that $\frac{|\nu_1'|}{\sqrt{\nu_2}}$ will not exceed t .

Each term in the expression for $F(\nu_2, |\nu_1'|)$ gives rise to a new term I_{ij} in the expression for $F(t)$

$$F(t) = I_{00} + \sum \sum \frac{C_{i, 2j}}{i! 2j!} I_{ij}.$$

The principal term I_{00} (corresponding to the case of a normal parent population) is in accordance with what »Student» has shown earlier

$$I_{00} = \frac{2\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{N-1}{2}\right)} [1 + t^2]^{-\frac{N}{2}},$$

which expression, as is known, has only one parameter, the number of the individuals in the sample.

Since, according to formula (24 a)

$$(-1)^i f_{\frac{N-1}{2}+i}^{(i)}(v_2) = \left(\frac{N}{2}\right)^i \left[f_{\frac{N-1}{2}+i}^{(i)}(v_2) - i f_{\frac{N-1}{2}+i-1}^{(i-1)}(v_2) + \dots (-1)^i f_{\frac{N-1}{2}}^{(0)}(v_2) \right]$$

(here and in the following exposition it is assumed that $\lambda_2 = 1$, that is, standardized characteristics are used in the coefficients C_{ij}), we find that

$$I_{i0} = \left(\frac{N}{2}\right)^i [f_i(t) - i f_{i-1}(t) + \dots (-1)^i f_0(t)],$$

where

$$f_i(t) = \frac{2\Gamma\left(\frac{N}{2} + i\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{N-1}{2} + i\right)} [1 + t^2]^{-\frac{N}{2}-i}.$$

This expression can also be written

$$I_{i0} = f_0(t) \cdot \left(\frac{N}{2}\right)^i \left[\frac{N+2i-2}{N+2i-3} \frac{N+2i-4}{N+2i-5} \dots \frac{N}{N-1} \left(\frac{1}{1+t^2}\right)^i - \right. \\ \left. - i \frac{N+2i-4}{N+2i-5} \dots \frac{N}{N-1} \left(\frac{1}{1+t^2}\right)^{i-1} + \dots (-1)^i \right].$$

If $t \sim \frac{1}{\sqrt{N}}$, we find that the parenthesis will be of the order N^{-i} and hence $I_{i0} \sim N^0$.

The expressions I_{0j} can be written as

$$I_{0j} = 2 \int_0^\infty f_{\frac{N-1}{2}}(v_2) H_{2j}(t\sqrt{v_2}) \varphi(t\sqrt{v_2}) \sqrt{v_2} dv_2.$$

After some operations we finally get

$$I_{0j} = (-1)^j N^j (N-1) [(N+1)(N+3) \dots (N+2j-3) f_j(t) - \\ - j(N-1)(N+1) \dots (N+2j-5) f_{j-1}(t) + \dots \\ \dots (-1)^j (N-2j+1)(N-2j+3) \dots (N-3) f_0(t)],$$

or also

$$\begin{aligned}
 I_{0j} = & (-1)^j 2^j [(N-1)(N+1) \dots (N+2j-3) I_{j0} + \\
 & + j(j-1) N(N-1) \dots (N+2j-5) I_{j-1,0} + \\
 & + \frac{j(j-1)^2(j-2)}{2!} N^2(N-1) \dots (N+2j-7) I_{j-2,0} + \\
 & + \frac{j(j-1)^2(j-2)^2(j-3)}{3!} N^3(N-1) \dots (N+2j-9) I_{j-3,0} + \dots \\
 & \dots + j! N^{j-1}(N-1) I_{10}],
 \end{aligned}$$

from which it immediately follows that I_{0j} is of the order N^j .

For the terms I_{ij} the following expressions are obtained

$$\begin{aligned}
 I_{ij} = & (-1)^j 2^j \left(\frac{N}{2}\right)^{i+j} \left\{ f_{i+j}(N+2i-1)(N+2i+1) \dots (N+2i+2j-3) - \right. \\
 & - f_{i+j-1}(N+2i-1)(N+2i+1) \dots (N+2i+2j-5) \left[i(N+2i-3) + \right. \\
 & + j(N+2i-1) \left. \right] + f_{i+j-2}(N+2i-3)(N+2i-1) \dots (N+2i+2j-7) \cdot \\
 & \cdot \left[\binom{i}{2}(N+2i-5) + ij(N+2i-3) + \binom{j}{2}(N+2i-1) \right] - \\
 & - f_{i+j-3}(N+2i-5)(N+2i-3) \dots (N+2i+2j-9) \left[\binom{i}{3}(N+2i-7) + \right. \\
 & + \binom{i}{2} \binom{j}{1}(N+2i-5) + \binom{i}{1} \binom{j}{2}(N+2i-3) + \binom{j}{3}(N+2i-1) \left. \right] + \dots \\
 & \left. \dots (-1)^{i+j} f_0(N-2j+1)(N-2j+3) \dots (N-3)(N-1) \right\}
 \end{aligned}$$

If t is of the order $\frac{1}{\sqrt{N}}$, these expressions are of the order N^j .

Hence the expressions I_{ij} will in general be of the order N^j .

Consequently we find that the terms

$$C_{20} I_{20}, C_{12} I_{11} \text{ and } C_{04} I_{02}$$

as well as the terms

$$C_{22} I_{21}, C_{14} I_{12} \text{ and } C_{06} I_{03}$$

will be of the order $\frac{1}{N}$.

If terms of the order $\frac{1}{N^2}$ are also to be taken into account, the following additional terms must be included

$$\begin{aligned} & C_{30} I_{30}, \\ & C_{40} I_{40}, C_{32} I_{31}, C_{24} I_{22}, C_{16} I_{13}, C_{08} I_{04}, \\ & C_{42} I_{41}, C_{34} I_{32}, C_{26} I_{23}, C_{18} I_{14}, C_{0,10} I_{05}, \\ & C_{44} I_{42}, C_{36} I_{33}, C_{28} I_{24}, C_{1,10} I_{15}, C_{0,12} I_{06}. \end{aligned}$$

The correction terms of order $\frac{1}{N}$ will be

$$\begin{aligned} & -\frac{\lambda_4}{4!} \frac{N-1}{N} [(2N+8)(f_2 - 2f_1 + f_0) + 8(f_1 - f_0)] - \\ & -\frac{40\lambda_3^2}{6!} \frac{N-1}{N} [(N+1)(N+3)(f_3 - 3f_2 + 3f_1 - f_0) - 6(f_1 - f_0)]. \end{aligned}$$

Hence the frequency distribution of $t = \frac{|\nu_1'|}{\sqrt{\nu_2}}$ can be written

$$\begin{aligned} F(t) = & f_0(t) - \frac{\lambda_4}{4!} \frac{N-1}{N} [(2N+8)(f_2 - 2f_1 + f_0) + 8(f_1 - f_0)] - \\ & - \frac{40\lambda_3^2}{6!} \frac{N-1}{N} [(N+1)(N+3)(f_3 - 3f_2 + 3f_1 - f_0) - 6(f_1 - f_0)] + \\ & + \text{terms of order } \frac{1}{N^2}. \end{aligned}$$

If the number of individuals in the sample is fairly large, the correction to be applied to »Students» formula will, therefore, be of little importance.

As the function $f_0(t)$ with growing N tends to the normal curve

$$\lim_{N \rightarrow \infty} \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{N-1}{2}\right)} [1 + t^2]^{-\frac{N}{2}} = \frac{\sqrt{N}}{\sqrt{2\pi}} e^{-\frac{N}{2} t^2} = \varphi(t),^1$$

there is reason to investigate how the two correction terms given above behave.

Assuming that t is of the order $\frac{1}{\sqrt{N}}$, and neglecting terms that are of the order $\frac{1}{N}$ in relation to the others, we find that

¹ Or rather $\frac{\sqrt{N-3}}{\sqrt{2\pi}} e^{-\frac{N-3}{2} t^2}$ as the second seminvariant $= \frac{1}{N-3}$.

$$\lim_{N \rightarrow \infty} N(f_1 - f_0) = \varphi(t)[1 - Nt^2];$$

$$\lim_{N \rightarrow \infty} N^2(f_2 - 2f_1 + f_0) = \varphi(t)[-1 - 2Nt^2 + N^2t^4];$$

$$\lim_{N \rightarrow \infty} N^3(f_3 - 2f_1 + f_0) = \varphi(t)[3 + 3Nt^2 + 3N^2t^4 - N^3t^6].$$

The correction terms will therefore be

$$-\frac{2\lambda_4}{N \cdot 4!} \varphi(t)[N^2t^4 - 6Nt^2 + 3] +$$

$$+\frac{40\lambda_3^2}{N \cdot 6!} \varphi(t)[N^3t^6 - 3N^2t^2 - 9Nt^2 - 3].$$

Putting $t\sqrt{N} = u$, we can write these expressions in the following form

$$\left[\text{where } \varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \right]$$

$$-\frac{2\lambda_4}{N \cdot 4!} \varphi(u)[u^4 - 6u^2 + 3] +$$

$$+\frac{40\lambda_3^2}{N \cdot 6!} \varphi(u)[(u^6 - 15u^4 + 45u^2 - 15) + 12(u^4 - 6u^2 + 3) + 18(u^2 - 1)].$$

The frequency function of $u = \frac{\sqrt{N}|r_1|}{\sqrt{r_2}}$ may therefore, for large values of N , be written in the following form

$$F(u) = 2 \left[\varphi(u) + \frac{\lambda_3^2}{N} \frac{\delta^2 \varphi}{\delta u^2} + \frac{1}{N} \left(\frac{2\lambda_3^2}{3} - \frac{\lambda_4}{12} \right) \frac{\delta^4 \varphi}{\delta u^4} + \frac{\lambda_3^2}{18N} \frac{\delta^6 \varphi}{\delta u^6} \right].$$

From this formula it will be easy to see how a large skewness and a large excess in the given parent population may disturb the formula given by »Student» for normal parent populations.

This correction will certainly be of rather small importance when the number of the individuals in the sample is large. For small numbers in the sample these correction terms may sometimes be of importance, but in most cases the formula given by »Student» may be valid also for non-normal parent populations.¹

¹ The correction terms of order $1/N$ given above would have been easily calculated ($N \leq 24$) from the tables given by »Student» (Biometrika 11. 1915—1917), if these tables had been calculated to five or six decimal places instead of only four.

CHAPTER 9.

The Correlation Distribution of the three Moments of the second Order in Samples from Normal Parent Populations.

A parent population, $\varphi(x, y)$, normally correlated, is given with its three seminvariants λ_{20} , λ_{02} and λ_{11} (or $r = \frac{\lambda_{11}}{\sqrt{\lambda_{20}\lambda_{02}}}$). As before it is postulated that the two variates x and y are measured from the means of the parent population, i. e. $\lambda_{10} = \lambda_{01} = 0$.

$$(63) \quad \varphi(x, y) = \frac{1}{2\pi \sqrt{\lambda_{20}\lambda_{02} - \lambda_{11}^2}} e^{-\frac{1}{2(1-r^2)} \left[\frac{x^2}{\lambda_{20}} + \frac{y^2}{\lambda_{02}} - 2r \frac{xy}{\sqrt{\lambda_{20}\lambda_{02}}} \right]}.$$

The characteristic function $U(t_1, t_2, t_3)$ of the three moments of the second order

$$v_{20}' = \frac{\sum x^2}{N}; \quad v_{02}' = \frac{\sum y^2}{N}; \quad v_{11}' = \frac{\sum xy}{N}$$

about the means of the parent population is obtained from the equation

$$(64 \text{ a}) \quad \begin{aligned} U(t_1, t_2, t_3) &= \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x, y) e^{\frac{x^2}{N} t_1 + \frac{y^2}{N} t_2 + \frac{xy}{N} t_3} dx dy \right]^N = \\ &= \left[\left(1 - \frac{2\lambda_{20}}{N} t_1 \right) \left(1 - \frac{2\lambda_{02}}{N} t_2 \right) - 4 \frac{\lambda_{11}^2}{N^2} t_1 t_2 - 2 \frac{\lambda_{11}}{N} t_3 - \frac{\lambda_{20}\lambda_{02} - \lambda_{11}^2}{N^2} t_3^2 \right]^{-\frac{N}{2}}. \end{aligned}$$

For the sake of simplicity the expression inside the parenthesis is hereafter denoted by D .

In deriving this formula it is postulated that t_1 , t_2 and t_3 satisfy such conditions that the integral is convergent and $D \neq 0$.

For imaginary values of t_1 , t_2 and t_3 these conditions are always satisfied.

For uncorrelated normal parent populations ($\lambda_{11} = 0$) we obtain

$$(64 \text{ b}) \quad U(t_1, t_2, t_3) = \left[\left(1 - 2 \frac{\lambda_{20}}{N} t_1 \right) \left(1 - 2 \frac{\lambda_{02}}{N} t_2 \right) - \frac{\lambda_{20} \lambda_{02}}{N^2} t_3^2 \right]^{-\frac{N}{2}}.$$

From these formulas (64 a, b) all the seminvariants of the distribution of the three second-order moments about the means of the parent population can be obtained without any great difficulty.

In this connection I shall not deal more fully with this formula, as it applies to the less important moments about the unknown means of the parent population.

The characteristic function of the distribution of the three central moments (taken about the means of the sample) of the second order can be obtained by first determining the characteristic function of the distribution of the zero moments of the first and second order, i. e. in a way quite analogous to the method employed in deriving the characteristic function of the distribution of the second moment.

The characteristic function of the distribution of these five moments

$$\nu'_{20} = \frac{\sum x^2}{N}; \quad \nu'_{02} = \frac{\sum y^2}{N}; \quad \nu'_{11} = \frac{\sum xy}{N}; \quad \nu'_{10} = \frac{\sum x}{N}; \quad \nu'_{01} = \frac{\sum y}{N}$$

becomes

$$U(t_1, t_2, t_3, t_4, t_5) = \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x, y) e^{\frac{x^2}{N} t_1 + \frac{y^2}{N} t_2 + \frac{xy}{N} t_3 + \frac{x}{N} t_4 + \frac{y}{N} t_5} dx dy \right]^N;$$

or

$$(65) \quad U(t_1, t_2, t_3, t_4, t_5) = D^{-\frac{N}{2}}.$$

$$D = \frac{\left(1 - \frac{2\tau_1}{N}\right) \left(1 - 2\frac{\tau_2}{N}\right)}{2ND} \left[\frac{\lambda_{20} t_4^2}{1 - \frac{2\tau_1}{N}} + \frac{\lambda_{02} t_5^2}{1 - 2\frac{\tau_2}{N}} + 2 \frac{\lambda_{20}^{1/2} \lambda_{02}^{1/2} \left(r + \frac{\tau_3}{N}\right) t_4 t_5}{\left(1 - 2\frac{\tau_1}{N}\right) \left(1 - 2\frac{\tau_2}{N}\right)} \right],$$

in which the following abbreviations have been used

$$\begin{aligned} \tau_1 &= t_1 \lambda_{20} (1 - r^2); \\ \tau_2 &= t_2 \lambda_{02} (1 - r^2); \\ \tau_3 &= t_3 \lambda_{20}^{1/2} \lambda_{02}^{1/2} (1 - r^2). \end{aligned}$$

The distribution $F(\nu'_{20}, \nu'_{02}, \nu'_{11}, \nu'_{10}, \nu'_{01})$ is then obtained by the »Inversion Theorem»

$$F = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(\omega_1 i, \omega_2 i, \omega_3 i, \omega_4 i, \omega_5 i) \cdot \\ \cdot e^{-\nu_{20}' \omega_1 i - \nu_{02}' \omega_2 i - \nu_{11}' \omega_3 i - \nu_{10}' \omega_4 i - \nu_{01}' \omega_5 i} d\omega_1 d\omega_2 d\omega_3 d\omega_4 d\omega_5.$$

The integral with regard to the two variables ω_4 and ω_5 may be considered as a normal correlation surface with the following parameters

$$L_{20} = \frac{\lambda_{20}}{N} \frac{\left(1 - 2 \frac{\tau_2}{N}\right)}{D}; \quad L_{02} = \frac{\lambda_{02}}{N} \frac{\left(1 - 2 \frac{\tau_1}{N}\right)}{D}; \\ L_{11} = \frac{\lambda_{20}^{1/2} \lambda_{02}^{1/2}}{N} \frac{\left(r + \frac{\tau_3}{N}\right)}{D}; \\ R = \frac{r + \frac{\tau_3}{N}}{\sqrt{\left(1 - 2 \frac{\tau_1}{N}\right) \left(1 - 2 \frac{\tau_2}{N}\right)}}; \quad 1 - R^2 = \frac{D(1 - r^2)}{\left(1 - 2 \frac{\tau_1}{N}\right) \left(1 - 2 \frac{\tau_2}{N}\right)}.$$

(In these formulas the nominations τ_1, τ_2 and τ_3 have been kept for simplicity instead of the corresponding imaginary variables.)

Thus the integration with regard to ω_4 and ω_5 results in

$$\frac{1}{2\pi \sqrt{L_{20} L_{02} (1 - R^2)}} e^{-\frac{1}{2(1-R^2)} \left[\frac{\nu_{10}'^2}{L_{20}} + \frac{\nu_{01}'^2}{L_{02}} - 2R \frac{\nu_{10}' \nu_{01}'}{\sqrt{L_{20} L_{02}}} \right]},$$

or with the values of L_{20} , L_{02} and R inserted

$$(66) \quad \frac{N}{2\pi \sqrt{\lambda_{20} \lambda_{02} (1 - r^2)}} e^{-\frac{N}{2(1-r^2)} \left[\frac{\nu_{10}'^2}{\lambda_{20}} + \frac{\nu_{01}'^2}{\lambda_{02}} - 2r \frac{\nu_{10}' \nu_{01}'}{\sqrt{\lambda_{20} \lambda_{02}}} \right]} \cdot D^{1/2} e^{\nu_{10}'^2 \omega_1 i + \nu_{01}'^2 \omega_2 i + \nu_{10}' \nu_{01}' \omega_3 i}.$$

We thus obtain

$$(67) \quad F(\nu_{20}', \nu_{02}', \nu_{11}', \nu_{10}', \nu_{01}') = \\ \varphi(\nu_{10}', \nu_{01}') \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D^{-\frac{N-1}{2}} e^{-(\nu_{20}' - \nu_{10}'^2) \omega_1 i - (\nu_{02}' - \nu_{01}'^2) \omega_2 i - (\nu_{11}' - \nu_{10}' \nu_{01}') \omega_3 i} d\omega_1 d\omega_2 d\omega_3.$$

As

$$\begin{aligned}v_{20}' - v_{10}'^2 &= v_{20}; \\v_{02}' - v_{01}'^2 &= v_{02}; \\v_{11}' - v_{10}' v_{01}' &= v_{11};\end{aligned}$$

we see that the three central moments of the second order vary independently of the two moments of the first order, though the moments of the same order are correlated to one another, and that the characteristic function of the correlation distribution of the former is

$$\begin{aligned}(68) \quad U(t_1, t_2, t_3) &= \left[\left(1 - 2 \frac{\lambda_{20}}{N} t_1\right) \left(1 - 2 \frac{\lambda_{02}}{N} t_2\right) - 4 \frac{\lambda_{11}^2}{N^2} t_1 t_2 - \right. \\&\quad \left. - 2 \frac{\lambda_{11}}{N} t_3 - \frac{\lambda_{20} \lambda_{02} - \lambda_{11}^2}{N^2} t_3^2 \right]^{-\frac{N-1}{2}}\end{aligned}$$

or the same function (64 a) as was shown for the distribution of the three zero moments, with the only difference that the exponent has been changed from $-\frac{N}{2}$ to $-\frac{N-1}{2}$, or exactly the same difference as was proved between the characteristic function of the zero moments and the central moment of the second order, considering one variate.

This implies that the only difference between the seminvariants of these two types of moments will be that they are in the same proportion to each other as N to $N-1$.

This characteristic function was obtained earlier in a somewhat different way by Romanowsky,¹ who also gave the formulas for the moments of the corresponding function.

Romanowsky also showed that this function is the characteristic function of R. A. Fisher's² previously found distribution of the three moments of the second order

$$\begin{aligned}(69) \quad F(v_{20}, v_{02}, v_{11}) &= \frac{1}{4\pi(N-3)!} \left[\frac{N^2}{\lambda_{20} \lambda_{02} - \lambda_{11}^2} \right]^{\frac{N-1}{2}} (v_{20} v_{02} - v_{11}^2)^{\frac{N-1}{2}} \\&\quad \cdot e^{-\frac{N}{2(1-r^2)} \left[\frac{v_{20}}{\lambda_{20}} + \frac{v_{02}}{\lambda_{02}} - 2r \frac{v_{11}}{\sqrt{\lambda_{20} \lambda_{02}}} \right]},\end{aligned}$$

¹ V. ROMANOWSKY. On the Moments of the Standard Deviation and of the Correlation Coefficient in Samples from normal. *Metron*. 5. 1925.

² R. A. FISHER. Frequency Distributions of the Values of the Correlation Coefficient in Samples from an indefinitely large Population.

Biometrika 10. 1915.

If in the characteristic function t_1 and t_2 are supposed to be $= 0$, the characteristic function of the distribution of ν_{11} , the correlation moment of the second order, is obtained

$$(70) \quad U(t_3) = \left[\left(1 - \frac{\lambda_{11}}{N} t_3 \right)^2 - \frac{\lambda_{20} \lambda_{02} t_3^2}{N^2} \right]^{-\frac{N-1}{2}} = \\ = \left[1 - t_3 \frac{\sqrt{\lambda_{20} \lambda_{02}} (1+r)}{N} \right]^{-\frac{N-1}{2}} \left[1 + t_3 \frac{\sqrt{\lambda_{20} \lambda_{02}} (1-r)}{N} \right]^{-\frac{N-1}{2}}.$$

The seminvariants of the distribution of this moment are easily obtained

$$(71) \quad L_n = \frac{N-1}{2} (n-1)! \frac{\lambda_{20}^{n/2} \lambda_{02}^{n/2}}{N^n} [(1+r)^n + (-1)^n (1-r)^n].$$

This formula has previously been given by Wishart and Bartlett,¹ but only in so far that they derived the corresponding formula for the seminvariants of ν_{11}' and concluded that the formula for ν_{11} must be of the same type and only differ by the factor $N-1$ instead of N .

Since the characteristic function for ν_{11} can be written as the product of two functions, the distribution of ν_{11} is interpretable as the distribution of a sum of two variates that are independent of each other. Both these variates are distributed according to a *Pearson's Type III*, one with the lower limit $= 0$ and with the parameters $b = \frac{N-1}{2}$ and $\gamma = \frac{(1+r)\sqrt{\lambda_{20}\lambda_{02}}}{N}$, the other with the upper limit $= 0$ (i. e. only capable of taking negative values) and with the parameters $b = \frac{N-1}{2}$ and $\gamma = -\frac{(1-r)\sqrt{\lambda_{20}\lambda_{02}}}{N}$. From this it follows that ν_{11} must vary between the limits $+\infty$ and $-\infty$.

In this connection I will not enter more fully upon the study of the distribution of ν_{11} , as this distribution has been thoroughly studied by several authors.²

¹ WISHART and BARTLETT. The Distribution of Second Order Moment Statistics in a Normal System. Proceedings of the Cambridge Philosophical Society. 28. 1932.

² KARL PEARSON, G. B. JEFFERY, ETHEL ELDERTON. On the Distribution of the First Product Moment-Coefficient in Samples drawn from an indefinitely large Normal Population. Biometrika. 21. 1929.

A. T. MCKAY. A Bessel Function Distribution. Biometrika. 24. 1932.

KARL PEARSON, S. A. STOUFFER, F. N. DAVID. Further Applications in Statistics of the $\tau_m(x)$ Bessel Function. Biometrika. 24. 1932.

KARL PEARSON. On the Applications of the Double Bessel Function to Statistical Problems. Biometrika. 25. 1933.

The moments of the distribution of the correlation coefficient $\varrho = \frac{r_{11}}{\sqrt{r_{20}r_{02}}}$ can be derived directly from the characteristic function of the distribution of the three moments of the second order.

r_{11} can for given values of r_{20} and r_{02} be regarded as a variate distributed according to a frequency function whose characteristics or moments depend on r_{20} and r_{02} .

The method demonstrated by Professor Wicksell of determining the moments $r_n'(x)$ of a dependent variate y as a function of the independent variate x can be readily extended to apply to several variates.

For two independent variates we get

$$(72) \quad f(x, y) r_n'(x, y) = \left(\frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 e^{-x\omega_1 i - y\omega_2 i} \left[\frac{\delta^n U(\omega_1 i, \omega_2 i, t_3)}{\delta t_3^n} \right]_{t_3=0}.$$

In this way the moments of r_{11} can be determined as functions of r_{20} and r_{02} .

Division by $\frac{1}{r_{20}r_{02}}^{n/2}$ then gives the moments of the correlation coefficient ϱ as functions of r_{20} and r_{02} , $r_n'(\varrho, r_{20}, r_{02})$.

$$\begin{aligned} f(r_{20}, r_{02}) r_n'(\varrho, r_{20}, r_{02}) &= \\ &= \left(\frac{1}{2\pi} \right)^2 \frac{1}{r_{20}r_{02}}^{n/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 e^{-r_{20}\omega_1 i - r_{02}\omega_2 i} \left[\frac{\delta^n U}{\delta t_3^n} \right]_{t_3=0}. \end{aligned}$$

The moments of the correlation coefficient are then found, independent of the values obtained for r_{20} and r_{02} , by integrating with regard to r_{20} and r_{02} over the whole range of variation

$$(73) \quad r_n'(\varrho) = \int_0^{\infty} \int_0^{\infty} dr_{20} dr_{02} \cdot \frac{1}{r_{20}r_{02}}^{n/2} \cdot \left(\frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 e^{-r_{20}\omega_1 i - r_{02}\omega_2 i} \cdot$$

$$\left[\frac{\delta^n U(\omega_1 i, \omega_2 i, t_3)}{\delta t_3^n} \right]_{t_3=0}.$$

For the first two moments of the correlation coefficient we obtain (if $\lambda_{20} = \lambda_{02} = 1$)

$$\begin{aligned} \left[\frac{\delta U}{\delta t_3} \right]_{t_3=0} &= \frac{N-1}{N} r \left[1 - 2 \frac{\omega_1 i}{N} - 2 \frac{\omega_2 i}{N} + 4 \frac{\omega_1 i \omega_2 i}{N^2} (1 - r^2) \right]^{-\frac{N+1}{2}}; \\ \left[\frac{\delta^2 U}{\delta t_3^2} \right]_{t_3=0} &= \frac{(N-1)(N+1)}{N^2} r^2 \left[1 - 2 \frac{\omega_1 i}{N} - 2 \frac{\omega_2 i}{N} + 4 \frac{\omega_1 i \omega_2 i}{N^2} (1 - r^2) \right]^{-\frac{N+3}{2}} + \\ &+ \frac{N-1}{N^2} (1 - r^2) \left[1 - 2 \frac{\omega_1 i}{N} - 2 \frac{\omega_2 i}{N} + 4 \frac{\omega_1 i \omega_2 i}{N^2} (1 - r^2) \right]^{-\frac{N+1}{2}}. \end{aligned}$$

Writing these expressions as the product of $\left(1 - \frac{2\omega_1 i}{N}\right)^{-\frac{N+1}{2}} \left(1 - \frac{2\omega_2 i}{N}\right)^{-\frac{N+1}{2}}$

and a power series of

$$\left(\frac{\omega_1 i}{1 - 2 \frac{\omega_1 i}{N}} \right) \left(\frac{\omega_2 i}{1 - 2 \frac{\omega_2 i}{N}} \right)$$

we get

$$\begin{aligned} v_1'(\varrho) &= \frac{N-1}{N} r \left[\int_0^\infty \frac{1}{v_{20}^{1/2}} f_{\frac{N+1}{2}}(v_{20}) dv_{20} \right]^2 + \\ &+ \sum_1^\infty \frac{1}{n!} \frac{\Gamma\left(\frac{N+1}{2} + n\right)}{\Gamma\left(\frac{N+1}{2}\right)} \cdot \left(\frac{4r^2}{N^2}\right)^n \left[\int_0^\infty \frac{1}{v_{20}^{1/2}} f_{\frac{N+1}{2} + n}^{(n)}(v_{20}) dv_{20} \right]^2; \end{aligned}$$

but

$$\int_0^\infty \frac{1}{v_{20}^{1/2}} f_{\frac{N+1}{2}}(v_{20}) dv_{20} = \frac{\left(\frac{N}{2}\right)^{1/2} \cdot \Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N+1}{2}\right)};$$

and

$$\begin{aligned} \int_0^\infty \frac{1}{v_{20}^{1/2}} f_{\frac{N+1}{2} + n}^{(n)}(v_{20}) dv_{20} &= (-1)^n \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdots \frac{2n-1}{2} \int_0^\infty \frac{1}{v_{20}^{n+1/2}} f_{\frac{N+1}{2} + n}(v_{20}) dv_{20} = \\ &= \frac{(2n)!}{2^n n!} \frac{N^{n+1/2} \Gamma\left(\frac{N}{2}\right)}{2^{n+1/2} \Gamma\left(\frac{N+1}{2} + n\right)}; \end{aligned}$$

and hence

$$\nu_1'(\varrho) = \frac{N-1}{N} r \cdot \frac{N}{2} \cdot \left[\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N+1}{2}\right)} \right]^2$$

(74 a)

$$\left[1 + \sum_1^{\infty} \frac{1}{n!} \cdot r^{2n} \cdot \left[\frac{(2n)!}{2^{2n} n!} \right]^2 \cdot \frac{\Gamma\left(\frac{N+1}{2}\right)}{\Gamma\left(\frac{N+1}{2} + n\right)} \right]$$

In the same way we obtain

$$\nu_2'(\varrho) = \frac{N-1}{N+1} r^2 \left[1 + \sum_1^{\infty} r^{2n} \cdot \frac{n! \Gamma\left(\frac{N+3}{2}\right)}{\Gamma\left(\frac{N+3}{2} + n\right)} \right] +$$

(74 b)

$$+ \frac{1-r^2}{N-1} \left[1 + \sum_1^{\infty} \frac{r^{2n} n! \Gamma\left(\frac{N+1}{2}\right)}{\Gamma\left(\frac{N+1}{2} + n\right)} \right]$$

CHAPTER 10.

The Characteristic Function of the Distribution of the three Moments of the Second Order for Random Samples from Non-Normal Parent Populations.

If the parent population cannot be described by means of the normal correlation distribution, but does not deviate too widely from this, we can make use of the so-called »Correlation Surface of Charlier's Type A«, which, in full analogy with what the case was in unidimensional distributions, represents the distribution in the form of a series, where the first term denotes the normal correlation surface and the following terms different derivatives of this, multiplied by certain constants.

$$(75 \text{ a}) \quad F(x, y) = \varphi(x, y) + \sum \sum (-1)^{i+j} \frac{A_{ij}}{i! j!} \frac{\delta^{i+j} \varphi(x, y)}{dx^i dy^j},$$

where

$$\varphi(x, y) = \frac{1}{2\pi \sqrt{ab - c^2}} e^{-\frac{1}{2(ab - c^2)} [bx^2 + ay^2 - 2cxy]}.$$

As before, it is assumed that the variates are measured from their means, that is $\lambda_{10} = \lambda_{01} = 0$.

If the three parameters a , b and c are chosen equal to the three seminvariants of the second order $\lambda_{20} = \sigma_x^2$; $\lambda_{02} = \sigma_y^2$; $\lambda_{11} = r\sigma_x \sigma_y$, the second-order coefficients vanish. (The first-order coefficients also vanish in consequence of the choice of means).

The relations between the A -coefficients and the seminvariants are given by the formula

$$(76 \text{ a}) \quad \sum_{i+j=3} \sum \frac{\lambda_{ij}}{i! j!} t_1^i t_2^j = 1 + \sum \sum_{i+j=3} \frac{A_{ij}}{i! j!} t_1^i t_2^j.$$

However, the law of distribution may also be written in the form

$$(75 \text{ b}) \quad F(x, y) = \varphi(x) \cdot \varphi(y) + \sum \sum (-1)^{i+j} \frac{A_{ij}'}{i! j!} \frac{\delta^i \varphi(x)}{\delta x^i} \cdot \frac{\delta^j \varphi(y)}{\delta y^j},$$

where

$$\varphi(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{x^2}{2\lambda_{20}}}, \quad \varphi(y) = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{y^2}{2\lambda_{02}}},$$

in which case A_{11}' does not vanish but becomes identical with λ_{11} .

The relations between the seminvariants and the A_{ij}' -coefficients will be

$$(76 \text{ b}) \quad e^{\lambda_{11} t_1 t_2 + \sum_{i+j=3} \sum \frac{\lambda_{ij}}{i! j!} t_1^i t_2^j} = 1 + A_{11}' t_1 t_2 + \sum_{i+j=3} \sum \frac{A_{ij}'}{i! j!} t_1^i t_2^j$$

We have

$$(77) \quad \begin{aligned} A_{10}' &= 0; \\ &\dots\dots\dots \\ A_{20}' &= 0; \\ A_{11}' &= \lambda_{11}; \\ &\dots\dots\dots \\ A_{30}' &= \lambda_{30} \\ A_{21}' &= \lambda_{21} \\ &\dots\dots\dots \\ A_{40}' &= \lambda_{40} \\ A_{31}' &= \lambda_{31} \\ A_{22}' &= \lambda_{22} + 2\lambda_{11}^2 \\ &\dots\dots\dots \\ A_{50}' &= \lambda_{50} \\ A_{41}' &= \lambda_{41} + 4\lambda_{30}\lambda_{11} \\ A_{32}' &= \lambda_{32} + 6\lambda_{21}\lambda_{11} \\ &\dots\dots\dots \\ A_{60}' &= \lambda_{60} + 10\lambda_{30}^2 \\ A_{51}' &= \lambda_{51} + 5\lambda_{40}\lambda_{11} + 10\lambda_{30}\lambda_{21} \\ A_{42}' &= \lambda_{42} + 8\lambda_{31}\lambda_{11} + 4\lambda_{30}\lambda_{12} + 6\lambda_{21}^2 \\ A_{33}' &= \lambda_{33} + 9\lambda_{22}\lambda_{11} + \lambda_{30}\lambda_{03} + 9\lambda_{21}\lambda_{12} + 6\lambda_{11}^3 \\ &\dots\dots\dots \\ A_{70}' &= \lambda_{70} + 35\lambda_{40}\lambda_{30} \\ A_{61}' &= \lambda_{61} + 6\lambda_{50}\lambda_{11} + 15\lambda_{40}\lambda_{21} + 20\lambda_{31}\lambda_{30} \\ A_{52}' &= \lambda_{52} + 10\lambda_{41}\lambda_{11} + 5\lambda_{40}\lambda_{12} + 20\lambda_{31}\lambda_{21} + 10\lambda_{22}\lambda_{30} + 20\lambda_{30}\lambda_{11}^2 \\ A_{43}' &= \lambda_{43} + 12\lambda_{32}\lambda_{11} + \lambda_{40}\lambda_{03} + 12\lambda_{31}\lambda_{12} + 18\lambda_{22}\lambda_{21} + 4\lambda_{13}\lambda_{30} + 36\lambda_{21}\lambda_{11}^2 \\ &\dots\dots\dots \\ A_{80}' &= \lambda_{80} + 56\lambda_{50}\lambda_{30} + 35\lambda_{40}^2; \\ A_{71}' &= \lambda_{71} + 7\lambda_{60}\lambda_{11} + 21\lambda_{50}\lambda_{21} + 35\lambda_{41}\lambda_{30} + 35\lambda_{40}\lambda_{31} + 70\lambda_{30}^2\lambda_{11} \\ A_{62}' &= \lambda_{62} + 12\lambda_{51}\lambda_{11} + 6\lambda_{50}\lambda_{12} + 30\lambda_{41}\lambda_{21} + 20\lambda_{32}\lambda_{30} + 15\lambda_{40}\lambda_{22} + 20\lambda_{31}^2 + \\ &\quad + 30\lambda_{40}\lambda_{11}^2 + 120\lambda_{30}\lambda_{21}\lambda_{11}; \end{aligned}$$

$$\begin{aligned}
 A_{53}' &= \lambda_{53} + 15\lambda_{42}\lambda_{11} + \lambda_{50}\lambda_{03} + 15\lambda_{41}\lambda_{12} + 30\lambda_{32}\lambda_{21} + 10\lambda_{23}\lambda_{30} + 5\lambda_{40}\lambda_{13} + \\
 &\quad + 30\lambda_{31}\lambda_{22} + 60\lambda_{31}\lambda_{11}^2 + 60\lambda_{30}\lambda_{12}\lambda_{11} + 90\lambda_{21}^2\lambda_{11}; \\
 (77) \quad A_{44}' &= \lambda_{44} + 16\lambda_{33}\lambda_{11} + 4\lambda_{41}\lambda_{03} + 24\lambda_{32}\lambda_{12} + 24\lambda_{23}\lambda_{21} + 4\lambda_{14}\lambda_{30} + \lambda_{40}\lambda_{04} + \\
 &\quad + 16\lambda_{31}\lambda_{13} + 18\lambda_{22}^2 + 72\lambda_{22}\lambda_{11}^2 + 16\lambda_{30}\lambda_{03}\lambda_{11} + 144\lambda_{21}\lambda_{12}\lambda_{11} + 24\lambda_{11}^4; \\
 &\quad \dots\dots\dots
 \end{aligned}$$

(If λ_{11} is put $= 0$ in these formulas we get the coefficients in the expression (75 a) for the correlation distribution.)

In the event of the variates being independent we have $\lambda_{ij} = 0$ if both i and $j \neq 0$, or, as a consequence, $A_{ij} = A_{i0} \cdot A_{0j}$.

If we assume that, as was the case in the unidimensional distribution, the two variates are generated by the influence of a great number (s) of elementary errors, certain (s_1) of which are common to both variates, then the seminvariants (or more correctly the standardized seminvariants) will be of the following order

$$\begin{aligned}
 \lambda_{ij} &\sim s^{1 - \frac{i+j}{2}} \\
 \lambda_{11} &\sim \frac{s_1}{s}.
 \end{aligned}$$

The order of magnitude of the A -coefficients in the first case (where λ_{11} is not included in the A -coefficients) will be

$$\begin{aligned}
 A_{ij} &\sim A_{i+j,0} \\
 A_{3n-4,0} &\sim A_{3n-2,0} \sim A_{3n,0} \sim s^{-\frac{n}{2}}
 \end{aligned}$$

and in the second case (where λ_{11} is included in the A -coefficients)

$$A_{ij}' \sim \lambda_{11}^j A_{i-j,0}'.$$

Particularly

$$A_{ii}' \sim \lambda_{11}^i \sim \left(\frac{s_1}{s}\right)^i.$$

If s_1 is small, i. e. the number of elementary errors common to both variates is small compared with the total number (s) of elementary errors, A_{ij}' will be of the same order of magnitude as $A_{i+j,0}$.

Therefore, when the last form of expansion (where λ_{11} is included in the A -coefficients) is employed, λ_{11} or more correctly r must be assumed to be rather small, so that the A -coefficients decline rapidly. If r is almost equal to 1 in amount, this last type of expansion has no practical application.

The different derivatives of $\varphi(x, y)$ can be written

$$\frac{\delta^{i+j}\varphi(x, y)}{\delta x^i \delta y^j} = (-1)^{i+j} H_{ij}(x, y) \varphi(x, y);$$

where $H_{ij}(x, y)$ is a polynomial of x and y of the degree $i + j$.

$$\text{If } u = \frac{1}{\sqrt{1-r^2}} \left(\frac{x}{\sigma_x} - r \frac{y}{\sigma_y} \right); \quad v = \frac{1}{\sqrt{1-r^2}} \left(\frac{y}{\sigma_y} - r \frac{x}{\sigma_x} \right);$$

$H_{ij}(x, y)$ can be written

$$(78) \quad \begin{aligned} H_{ij}(x, y) = & H_i(u) H_j(v) + \frac{r}{\sigma_x \sigma_y (1-r^2)} \frac{i \cdot j}{1!} H_{i-1}(u) H_{j-1}(v) + \\ & + \left[\frac{r}{\sigma_x \sigma_y (1-r^2)} \right]^2 \frac{i(i-1)j(j-1)}{2!} H_{i-2}(u) H_{j-2}(v) + \dots, \end{aligned}$$

where

$$H_i(u) = \left[\frac{1}{\sigma_x \sqrt{1-r^2}} \right]^i \left[u^i - \frac{i(i-1)}{2} u^{i-2} + \dots \right].$$

For $r=0$, $H_{ij}(x, y)$ splits direct into the product $H_i(x) H_j(y)$.

Besides the two variates x and y , the polynomials contain three parameters λ_{20} , λ_{02} and λ_{11} (or σ_x , σ_y and r).

The integral $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{ij}(x, y) \varphi(x, y) dx dy$ is always exactly equal to 0. But if for $\varphi(x, y)$ we substitute another normal correlation distribution $\psi(x, y)$ with other parameters L_{20} , L_{02} and L_{11} , we get integrals whose value will not be exactly equal to zero if the degree $i+j$ is even.

$$(79 \text{ a}) \quad \begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{2i, 2j}(x, y) \psi(x, y) dx dy = & \left[\frac{1}{1-r^2} \right]^{2i+2j} \frac{2i! 2j!}{2^{i+j} i! j!} \left[a^i b^j + \frac{i \cdot j}{2!} a^{i-1} b^{j-1} c^2 + \right. \\ & \left. + \frac{i(i-1)j(j-1)}{4!} a^{i-2} b^{j-2} c^4 + \dots + \frac{i(i-1) \dots (i-j+1)j!}{2j!} a^{i-j} c^{2j} \right] \end{aligned}$$

and

$$(79 \text{ b}) \quad \begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{2i+1, 2j+1}(x, y) \psi(x, y) dx dy = \\ & - \left[\frac{1}{1-r^2} \right]^{2i+2j+2} \frac{(2i+1)! (2j+1)!}{2^{i+j+1} i! j!} \left[a^i b^j c + \frac{i \cdot j}{3!} a^{i-1} b^{j-1} c^3 + \right. \\ & \left. + \frac{i(i-1)j(j-1)}{5!} a^{i-2} b^{j-2} c^5 + \dots + \frac{i(i-1) \dots (i-j+1)j!}{(2j+1)!} a^{i-j} c^{2j+1} \right]. \end{aligned}$$

(Formulas reminiscent of the general formulas for the different moments of the normal correlation surface) and where

$$\begin{aligned}
 a &= L_{20} - \lambda_{20} + \frac{\lambda_{11}^2}{\lambda_{02}^2} \left[L_{02} - \lambda_{02} \right] - 2 \frac{\lambda_{11}}{\lambda_{02}} \left[L_{11} - \lambda_{11} \right], \\
 b &= L_{02} - \lambda_{02} + \frac{\lambda_{11}^2}{\lambda_{20}^2} \left[L_{20} - \lambda_{20} \right] - 2 \frac{\lambda_{11}}{\lambda_{20}} \left[L_{11} - \lambda_{11} \right], \\
 c &= 2 \left[\left(1 + \frac{\lambda_{11}^2}{\lambda_{20} \lambda_{02}} \right) \left(L_{11} - \lambda_{11} \right) - \frac{\lambda_{11}}{\lambda_{20}} \left(L_{20} - \lambda_{20} \right) - \frac{\lambda_{11}}{\lambda_{02}} \left(L_{02} - \lambda_{02} \right) \right].
 \end{aligned}$$

We shall now determine the characteristic function $U(t_1, t_2, t_3)$ of the distribution of the three zero moments of the second order ν_{20}' , ν_{02}' and ν_{11}' if a random sample of N individuals is taken.

(For simplicity's sake it is hereafter assumed that $\lambda_{20} = \lambda_{02} = 1$, that is, standardized variates are used for the parent distribution).

$$\begin{aligned}
 (80) \quad U(t_1, t_2, t_3) &= \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) e^{\frac{x^2}{N}t_1 + \frac{y^2}{N}t_2 + \frac{xy}{N}t_3} dx dy \right]^N \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) e^{\frac{x^2}{N}t_1 + \frac{y^2}{N}t_2 + \frac{xy}{N}t_3} dx dy + \\
 &\quad + \sum \sum \frac{A_{ij}}{i!j!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{ij}(x, y) \varphi(x, y) e^{\frac{x^2}{N}t_1 + \frac{y^2}{N}t_2 + \frac{xy}{N}t_3} dx dy.
 \end{aligned}$$

As previously shown, the first integral is

$$\left[1 - 2 \frac{t_1}{N} - 2 \frac{t_2}{N} + 4 \frac{t_1 t_2}{N^2} (1 - r^2) - 2r \frac{t_3}{N} - \frac{t_3^2}{N^2} (1 - r^2) \right]^{-1/2} = D^{-1/2}.$$

The expression $\varphi(x, y) e^{\frac{x^2}{N}t_1 + \frac{y^2}{N}t_2 + \frac{xy}{N}t_3}$ is, except for the factor $D^{-1/2}$, interpretable as a normal correlation function with the seminvariants

$$\begin{aligned}
 L_{20} &= \frac{1 - 2 \frac{t_2}{N} (1 - r^2)}{D}; & L_{02} &= \frac{1 - 2 \frac{t_1}{N} (1 - r^2)}{D}; \\
 L_{11} &= \frac{r + \frac{t_3}{N} (1 - r^2)}{D};
 \end{aligned}$$

and for the integral

$$I_{ij} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{ij}(x, y) \varphi(x, y) e^{\frac{x^2}{N}t_1 + \frac{y^2}{N}t_2 + \frac{xy}{N}t_3} dx dy;$$

we obtain, in accordance with what has been said before (79 a, b),

$$I_{2i, 2j+1} = 0; \quad I_{2i+1, 2j} = 0;$$

$$(81) \quad I_{2i, 2j} = \frac{2i! 2j!}{N^{i+j}} \left[\frac{\tau_1^i \tau_2^j}{i! j!} + \frac{\tau_1^{i-1}}{(i-1)!} \cdot \frac{\tau_2^{j-1}}{(j-1)!} \cdot \frac{\tau_3^2}{2!} + \dots \right] D^{-1/2}$$

$$I_{2i+1, 2j+1} = \frac{(2i+1)! (2j+1)!}{N^{i+j+1}} \left[\frac{\tau_1^i \tau_2^j}{i! j!} \cdot \frac{\tau_3}{1!} + \frac{\tau_1^{i-1}}{(i-1)!} \cdot \frac{\tau_2^{j-1}}{(j-1)!} \cdot \frac{\tau_3^3}{3!} + \dots \right] D^{-1/2},$$

where (as everywhere in the following)

$$(82 \text{ a}) \quad \tau_1 = \frac{t_1 - \frac{1}{N} \left[2t_1 t_2 - \frac{1}{2} t_3^2 \right]}{D}, \quad \tau_2 = \frac{t_2 - \frac{1}{N} \left[2t_1 t_2 - \frac{1}{2} t_3^2 \right]}{D},$$

$$\tau_3 = \frac{t_3 + \frac{1}{N} \left[4t_1 t_2 - t_3^2 \right]}{D}.$$

If λ_{20} and λ_{02} are not equal to 1 we have

$$(82 \text{ b}) \quad \tau_1 = \frac{t_1 - \frac{\lambda_{02}}{N} \left[2t_1 t_2 - \frac{1}{2} t_3^2 \right]}{D}, \quad \tau_2 = \frac{t_2 - \frac{\lambda_{20}}{N} \left[2t_1 t_2 - \frac{1}{2} t_3^2 \right]}{D},$$

$$\tau_3 = \frac{t_3 + \frac{\lambda_{11}}{N} \left[4t_1 t_2 - t_3^2 \right]}{D}.$$

Thus the expression $U(t_1, t_2, t_3)$ will be

$$(83) \quad U(t_1, t_2, t_3) = D^{-\frac{N}{2}} \left[1 + \sum \sum \sum \sum \frac{A_{2i+k, 2j+k}}{N^{i+j+k} i! j! k!} \tau_1^i \tau_2^j \tau_3^k \right]^N$$

or

$$U(t_1, t_2, t_3) = D^{-\frac{N}{2}} \left[1 + \sum \sum \sum \sum \frac{B_{ijk}}{i! j! k!} \tau_1^i \tau_2^j \tau_3^k \right]$$

and the connection between the A_{ijk} and B_{ijk} coefficients is easy to understand.

For the coefficients B_{ijk} up to the fourth order we obtain

$$\begin{aligned}
 B_{100} &= 0 \\
 &\dots\dots\dots \\
 B_{200} &= \frac{1}{N} \lambda_{40}; \\
 B_{110} &= \frac{1}{N} [\lambda_{22} + 2\lambda_{11}^2]; \\
 &\dots\dots\dots \\
 B_{300} &= \frac{1}{N^2} [\lambda_{60} + 10\lambda_{30}^2]; \\
 B_{210} &= \frac{1}{N^2} [\lambda_{42} + 6\lambda_{21}^2 + 4\lambda_{30}\lambda_{12} + 8\lambda_{31}\lambda_{11}]; \\
 &\dots\dots\dots \\
 B_{400} &= \frac{1}{N^3} [\lambda_{80} + 56\lambda_{50}\lambda_{30} + 35\lambda_{40}^2] + 3\frac{N-1}{N^3} \lambda_{40}^2; \\
 B_{310} &= \frac{1}{N^3} [\lambda_{62} + 12\lambda_{51}\lambda_{11} + 6\lambda_{50}\lambda_{12} + 30\lambda_{41}\lambda_{21} + 20\lambda_{32}\lambda_{30} + 15\lambda_{40}\lambda_{22} + 20\lambda_{31}^2 + \\
 &\quad + 30\lambda_{40}\lambda_{11}^2 + 120\lambda_{30}\lambda_{21}\lambda_{11}] + \frac{N-1}{N^3} [3\lambda_{40}\lambda_{22} + 6\lambda_{40}\lambda_{11}^2]; \\
 B_{220} &+ \frac{1}{N^3} [\lambda_{44} + 16\lambda_{33}\lambda_{11} + 4\lambda_{41}\lambda_{03} + 24\lambda_{32}\lambda_{12} + 24\lambda_{23}\lambda_{21} + 4\lambda_{14}\lambda_{30} + \lambda_{40}\lambda_{04} + \\
 (84) \quad &\quad + 16\lambda_{31}\lambda_{13} + 18\lambda_{22}^2 + 72\lambda_{22}\lambda_{11}^2 + 16\lambda_{30}\lambda_{03}\lambda_{11} + 144\lambda_{21}\lambda_{12}\lambda_{11} + 24\lambda_{11}^4] + \\
 &\quad + \frac{N-1}{N^3} [\lambda_{40}\lambda_{04} + 2\lambda_{22}^2 + 8\lambda_{22}\lambda_{11}^2 + 8\lambda_{11}^4]; \\
 &\dots\dots\dots \\
 &\dots\dots\dots \\
 B_{001} &= \lambda_{11}; \\
 &\dots\dots\dots \\
 B_{101} &= \frac{1}{N} \lambda_{31}; \\
 &\dots\dots\dots \\
 B_{201} &= \frac{1}{N^2} [\lambda_{51} + 10\lambda_{30}\lambda_{21} + 5\lambda_{40}\lambda_{11}] + \frac{N-1}{N^2} \lambda_{40}\lambda_{11}; \\
 B_{111} &= \frac{1}{N^2} [\lambda_{33} + 9\lambda_{22}\lambda_{11} + \lambda_{30}\lambda_{03} + 9\lambda_{21}\lambda_{12} + 6\lambda_{11}^3] + \\
 &\quad + \frac{N-1}{N^2} [\lambda_{22}\lambda_{11} + 2\lambda_{11}^3]; \\
 &\dots\dots\dots \\
 B_{301} &= \frac{1}{N^3} [\lambda_{71} + 7\lambda_{60}\lambda_{11} + 21\lambda_{50}\lambda_{21} + 35\lambda_{41}\lambda_{30} + 35\lambda_{40}\lambda_{31} + 70\lambda_{30}^2\lambda_{11}] + \\
 &\quad + \frac{N-1}{N^3} [\lambda_{60}\lambda_{11} + 3\lambda_{40}\lambda_{31} + 10\lambda_{30}^2\lambda_{11}];
 \end{aligned}$$

$$B_{211} = \frac{1}{N^3} [\lambda_{53} + 15\lambda_{42}\lambda_{11} + \lambda_{50}\lambda_{03} + 15\lambda_{41}\lambda_{12} + 30\lambda_{32}\lambda_{21} + 10\lambda_{23}\lambda_{30} + 5\lambda_{40}\lambda_{13} + \\ + 30\lambda_{31}\lambda_{22} + 60\lambda_{31}\lambda_{11}^2 + 60\lambda_{30}\lambda_{12}\lambda_{11} + 90\lambda_{21}^2\lambda_{11}] + \frac{N-1}{N^3} [\lambda_{42}\lambda_{11} + \\ + \lambda_{40}\lambda_{13} + 2\lambda_{31}\lambda_{22} + 12\lambda_{31}\lambda_{11}^2 + 4\lambda_{30}\lambda_{12}\lambda_{11} + 6\lambda_{21}^2\lambda_{11}];$$

.....

.....

$$B_{092} = \frac{1}{N} [\lambda_{22} + 2\lambda_{11}^2] + \frac{N-1}{N} \lambda_{11}^2;$$

.....

$$B_{102} = \frac{1}{N^2} [\lambda_{42} + 8\lambda_{31}\lambda_{11} + 4\lambda_{30}\lambda_{12} + 6\lambda_{21}^2] + 2 \frac{N-1}{N^2} \lambda_{31}\lambda_{11};$$

.....

$$B_{202} = \frac{1}{N^3} [\lambda_{62} + 12\lambda_{51}\lambda_{11} + 6\lambda_{50}\lambda_{12} + 30\lambda_{41}\lambda_{21} + 20\lambda_{32}\lambda_{30} + 15\lambda_{40}\lambda_{22} + 20\lambda_{31}^2 + \\ + 30\lambda_{40}\lambda_{11}^2 + 120\lambda_{30}\lambda_{21}\lambda_{11}] + \frac{N-1}{N^3} [2\lambda_{51}\lambda_{11} + \lambda_{40}\lambda_{22} + 2\lambda_{31}^2 + \\ + 12\lambda_{40}\lambda_{11}^2 + 20\lambda_{30}\lambda_{21}\lambda_{11}] + \frac{(N-1)(N-2)}{N^3} \lambda_{40}\lambda_{11}^2;$$

$$(84) \quad B_{112} = \frac{1}{N^3} [\lambda_{44} + 16\lambda_{33}\lambda_{11} + 4\lambda_{41}\lambda_{03} + 24\lambda_{32}\lambda_{12} + 24\lambda_{23}\lambda_{21} + 4\lambda_{14}\lambda_{30} + \lambda_{40}\lambda_{04} + \\ + 16\lambda_{31}\lambda_{13} + 18\lambda_{22}^2 + 72\lambda_{22}\lambda_{11}^2 + 16\lambda_{30}\lambda_{03}\lambda_{11} + 144\lambda_{21}\lambda_{12}\lambda_{11} + 24\lambda_{11}^4] + \\ + \frac{N-1}{N^3} [2\lambda_{33}\lambda_{11} + \lambda_{22}^2 + 2\lambda_{31}\lambda_{13} + 22\lambda_{22}\lambda_{11}^2 + 2\lambda_{30}\lambda_{03}\lambda_{11} + \\ + 18\lambda_{21}\lambda_{12}\lambda_{11} + 16\lambda_{11}^4] + \frac{(N-1)(N-2)}{N^3} [\lambda_{22}\lambda_{11} + 2\lambda_{11}^4];$$

.....

$$B_{003} = \frac{1}{N^2} [\lambda_{33} + 9\lambda_{22}\lambda_{11} + \lambda_{30}\lambda_{03} + 9\lambda_{21}\lambda_{12} + 6\lambda_{11}^3] + \frac{N-1}{N^2} [3\lambda_{22}\lambda_{11} + 6\lambda_{11}^3] + \\ + \frac{(N-1)(N-2)}{N^2} \lambda_{11}^3;$$

.....

$$B_{103} = \frac{1}{N^3} [\lambda_{53} + 15\lambda_{42}\lambda_{11} + \lambda_{50}\lambda_{03} + 15\lambda_{41}\lambda_{12} + 30\lambda_{32}\lambda_{21} + 10\lambda_{23}\lambda_{30} + 5\lambda_{40}\lambda_{13} + \\ + 30\lambda_{31}\lambda_{22} + 60\lambda_{31}\lambda_{11}^2 + 60\lambda_{30}\lambda_{12}\lambda_{11} + 90\lambda_{21}^2\lambda_{11}] + \frac{N-1}{N^3} [3\lambda_{42}\lambda_{11} + 3\lambda_{31}\lambda_{22} + \\ + 30\lambda_{31}\lambda_{11}^2 + 12\lambda_{30}\lambda_{12}\lambda_{11} + 18\lambda_{21}^2\lambda_{11}] + 3 \frac{(N-1)(N-2)}{N^3} \lambda_{31}\lambda_{11}^2;$$

.....

.....

$$\begin{aligned}
 B_{004} = & \frac{1}{N^3} [\lambda_{44} + 16\lambda_{33}\lambda_{11} + 4\lambda_{41}\lambda_{03} + 24\lambda_{32}\lambda_{12} + 24\lambda_{23}\lambda_{21} + 4\lambda_{14}\lambda_{30} + \lambda_{40}\lambda_{04} + \\
 & + 16\lambda_{31}\lambda_{13} + 18\lambda_{22}^2 + 72\lambda_{22}\lambda_{11}^2 + 16\lambda_{30}\lambda_{03}\lambda_{11} + 144\lambda_{21}\lambda_{12}\lambda_{11} + 24\lambda_{11}^4] + \\
 & + \frac{N-1}{N^3} [4\lambda_{33}\lambda_{11} + 3\lambda_{22}^2 + 48\lambda_{22}\lambda_{11}^2 + 4\lambda_{30}\lambda_{03}\lambda_{11} + 36\lambda_{21}\lambda_{12}\lambda_{11} + \\
 (84) \quad & + 36\lambda_{11}^4] + \frac{(N-1)(N-2)}{N^3} [6\lambda_{22}\lambda_{11}^2 + 12\lambda_{11}^4] + \\
 & + \frac{(N-1)(N-2)(N-3)}{N^3} \lambda_{11}^4; \\
 & \dots\dots\dots
 \end{aligned}$$

These formulas have been calculated under the assumption that the first correlation seminvariant λ_{11} is included in the A -coefficients (according to formula 77), although it has also been assumed that λ_{11} or r is included in the main term. This has been done here to enable a choice to be made later on of whichever of these two expressions is found to be of most use. If this choice has been made, we have only to put λ_{11} or r equal to zero either in the coefficients B_{ijk} or in the expressions D , τ_1 , τ_2 , τ_3 involved in the characteristic function.

In this connection I shall not go into the determination of the corresponding multiple frequency distribution, but proceed to the determination of the characteristic function of the three central moments of the second order.

As before, it is simplest first of all to determine the characteristic function $U(t_1, t_2, t_3, t_4, t_5)$ of the distribution of the three second-order moments about zero and of the two moments of the first order.

$$(85) \quad U = \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) e^{\frac{x^2}{N}t_1 + \frac{y^2}{N}t_2 + \frac{xy}{N}t_3 + \frac{x}{N}t_4 + \frac{y}{N}t_5} dx dy \right]^N.$$

Thus we have to evaluate the integrals

$$I_{ij} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{ij} \varphi(x, y) e^{\frac{x^2}{N}t_1 + \frac{y^2}{N}t_2 + \frac{xy}{N}t_3 + \frac{x}{N}t_4 + \frac{y}{N}t_5} dx dy.$$

The integral I_{00} is given before in formula (65) = U_0 .

By linear substitution

$$x = x' + a; \quad y = y' + b,$$

where

$$a = \frac{1}{N} \frac{t_4 \left[1 - \frac{2t_2(1-r^2)}{N} \right] + t_5 \left[r + \frac{t_3(1-r^2)}{N} \right]}{D}$$

$$b = \frac{1}{N} \frac{t_4 \left[r + \frac{t_3(1-r^2)}{N} \right] + t_5 \left[1 - \frac{2t_1(1-r^2)}{N} \right]}{D},$$

the integral I_{ij} is converted into

$$(86) \quad I_{ij} = U_0 \iint H_{ij}(x+a, y+b) \frac{D^{1/2}}{2\pi\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)} \left[x^2 + y^2 - 2rxy \right] + \frac{x^2}{N} t_1 + \frac{y^2}{N} t_2 + \frac{xy}{N} t_3} dx dy.$$

But $H_{ij}(x+a, y+b)$ can be expressed by aid of the following formula in terms of the polynomials $H_{ij}(x, y)$

$$(87) \quad H_{ij}(x+a, y+b) = H_{ij}(x, y) + [a_1 i H_{i-1, j}(x, y) + b_1 j H_{i, j-1}(x, y)] +$$

$$+ \left[a_1^2 \frac{i(i-1)}{2!} H_{i-2, j}(x, y) + a_1 b_1 i j H_{i-1, j-1}(x, y) + b_1^2 \frac{j(j-1)}{2!} H_{i, j-2}(x, y) \right] + \dots$$

$$\dots + a_1^i b_1^j H_{00}(x, y)$$

where

$$a_1 = \frac{a - rb}{1 - r^2}; \quad b_1 = \frac{b - ra}{1 - r^2};$$

or

$$(88) \quad a_1 = \frac{\tau_4}{N} = \frac{t_4 \left(1 - \frac{2t_2}{N} - \frac{rt_3}{N} \right) + t_5 \left(\frac{2rt_1}{N} + \frac{t_3}{N} \right)}{ND}$$

$$b_1 = \frac{\tau_5}{N} = \frac{t_4 \left(\frac{2rt_2}{N} + \frac{t_3}{N} \right) + t_5 \left(1 - \frac{2t_1}{N} - \frac{rt_3}{N} \right)}{ND}.$$

If we now make use of the previously deduced expressions for the integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{ij}(x, y) \varphi(x, y) e^{\frac{x^2}{N} t_1 + \frac{y^2}{N} t_2 + \frac{xy}{N} t_3} dx dy,$$

a simple regrouping will give

$$(89) \quad \begin{aligned} U &= U_0 \left[1 + \sum \sum \sum \sum \sum \frac{A_{2i+k+l, 2j+k+m}}{N^{i+j+k+l+m} i! j! k! l! m!} \tau_1^i \tau_2^j \tau_3^k \tau_4^l \tau_5^m \right]^N = \\ &= U_0 \left[1 + \sum \sum \sum \sum \sum \frac{B_{ijklm}}{i! j! k! l! m!} \tau_1^i \tau_2^j \tau_3^k \tau_4^l \tau_5^m \right], \end{aligned}$$

where U_0 denotes the characteristic function that corresponds to a normal parent population, and where $\tau_1, \tau_2, \tau_3, \tau_4$ and τ_5 denote the expressions given in formula (82 a) and formula (88).

As is seen from the formula (65), U_0 is composed of two expressions, the first of which contains only t_1, t_2 and t_3 , while the second contains t_4 and t_5 multiplied by certain functions of t_1, t_2 and t_3 . In the same way τ_1, τ_2, τ_3 contain only t_1, t_2 and t_3 , while τ_4 and τ_5 contain all the five parameters t .

From this characteristic function we derive the characteristic function for the distribution of the three central moments of the second order by a process fully analogous to that used in the case of normal correlation distribution.

The distribution of the five moments of the first and second order about zero is given by the »Inversion Theorem».

$$\begin{aligned} F(\nu_{20}', \nu_{02}', \nu_{11}', \nu_{10}', \nu_{01}') &= \\ &= \left(\frac{1}{2\pi} \right)^5 \int \int \int \int \int_{-\infty}^{\infty} U e^{-\nu_{20}' \omega_1 i - \nu_{02}' \omega_2 i - \nu_{11}' \omega_3 i - \nu_{10}' \omega_4 i - \nu_{01}' \omega_5 i} d\omega_1 d\omega_2 d\omega_3 d\omega_4 d\omega_5. \end{aligned}$$

The distribution of the three central moments ν_{20}, ν_{02} and ν_{11} may be obtained by first making the substitutions

$$\nu_{20}' = \nu_{20} + \nu_{10}'^2; \quad \nu_{02}' = \nu_{02} + \nu_{01}'^2; \quad \nu_{11}' = \nu_{11} + \nu_{10}' \nu_{01}';$$

and then integrating with respect to ν_{10}' and ν_{01}'

$$F(\nu_{20}, \nu_{02}, \nu_{11}) = \int \int_{-\infty}^{\infty} F(\nu_{20} + \nu_{10}'^2, \nu_{02} + \nu_{01}'^2, \nu_{11} + \nu_{10}' \nu_{01}', \nu_{10}', \nu_{01}') d\nu_{10}' d\nu_{01}'$$

or

$$(90) \quad \begin{aligned} F(\nu_{20}, \nu_{02}, \nu_{11}) &= \left(\frac{1}{2\pi} \right)^3 \int \int \int_{-\infty}^{\infty} d\omega_1 d\omega_2 d\omega_3 \left[e^{-\nu_{20} \omega_1 i - \nu_{02} \omega_2 i - \nu_{11} \omega_3 i} \cdot \right. \\ &\quad \left. \int \int_{-\infty}^{\infty} d\nu_{10}' d\nu_{01}' e^{-\nu_{10}'^2 \omega_4 i - \nu_{01}'^2 \omega_5 i - \nu_{10}' \nu_{01}' \omega_3 i} \cdot \left(\frac{1}{2\pi} \right)^2 \int \int_{-\infty}^{\infty} U e^{-\nu_{10}' \omega_4 i - \nu_{01}' \omega_5 i} d\omega_4 d\omega_5 \right]. \end{aligned}$$

That is to say, the characteristic function of the distribution of the central moments of the second order will be

$$\begin{aligned}
 (91) \quad U(t_1, t_2, t_3) &= \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu_{10}' d\nu_{01}' e^{-\nu_{10}'^2 t_1 - \nu_{01}'^2 t_2 - \nu_{10}' \nu_{01}' t_3} \left(\frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U e^{-\nu_{10}' \omega_4 i - \nu_{01}' \omega_5 i} d\omega_4 d\omega_5 \\
 \text{or} \\
 U(t_1, t_2, t_3) &= \\
 &= D^{-\frac{N}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu_{10}' d\nu_{01}' e^{-\nu_{10}'^2 t_1 - \nu_{01}'^2 t_2 - \nu_{10}' \nu_{01}' t_3} \left(\frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{[\dots]} e^{-\nu_{10}' \omega_4 i - \nu_{01}' \omega_5 i} d\omega_4 d\omega_5 + \\
 &\quad + D^{-\frac{N}{2}} \sum \sum \sum \sum \sum \frac{B_{ijklm}}{i! j! k! l! m!} \tau_1^i \tau_2^j \tau_3^k \times \\
 &\quad \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu_{10}' d\nu_{01}' e^{-\nu_{10}'^2 t_1 - \nu_{01}'^2 t_2 - \nu_{10}' \nu_{01}' t_3} \left(\frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\omega_4' i)^l (\omega_5' i)^m e^{[\dots]} e^{-\nu_{10}' \omega_4 i - \nu_{01}' \omega_5 i} d\omega_4 d\omega_5,
 \end{aligned}$$

where

$$\begin{aligned}
 \omega_4' &= \frac{\omega_4 \left[1 - \frac{2t_2}{N} - \frac{rt_3}{N} \right] + \omega_5 \left[\frac{2rt_1}{N} + \frac{t_3}{N} \right]}{D} \\
 \omega_5' &= \frac{\omega_4 \left[\frac{2rt_2}{N} + \frac{t_3}{N} \right] + \omega_5 \left[1 - \frac{2t_1}{N} - \frac{rt_3}{N} \right]}{D}
 \end{aligned}$$

and where $e^{[\dots]}$ denotes the exponential expression given in formula (65).

Since

$$\begin{aligned}
 &\left(\frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{[\dots]} e^{-\nu_{10}' \omega_4 i - \nu_{01}' \omega_5 i} d\omega_4 d\omega_5 = \varphi(\nu_{10}', \nu_{01}') = \\
 &= D^{1/2} \frac{1}{2\pi \sqrt{1-r^2}} e^{-\frac{N}{2(1-r^2)} [\nu_{10}'^2 + \nu_{01}'^2 - 2r\nu_{10}'\nu_{01}'] + \nu_{10}'^2 t_1 + \nu_{01}'^2 t_2 + \nu_{10}' \nu_{01}' t_3}
 \end{aligned}$$

the first integral, as was already shown, will be $D^{1/2}$.

In the other integrals I_{lm} after the first integration, the last part may be written as a series of derivatives of the power $l+m$ of the correlation distribution $\varphi(\nu_{10}', \nu_{01}')$ given above, or as a product of a series of Hermitian polynomials, all of the degree $l+m$, with the above correlation distribution $\varphi(\nu_{10}', \nu_{01}')$.

If these expressions are multiplied by $e^{-\nu_{10}'^2 t_1 - \nu_{01}'^2 t_2 - \nu_{10}' \nu_{01}' t_3}$ and thereafter integrated with regard to ν_{10}' and ν_{01}' , we finally get

$$\frac{I_{2l, 2m}}{2l! 2m!} = (-1)^{l+m} D^{1/2} \left[\frac{\tau_1^l}{l!} \cdot \frac{\tau_2^m}{m!} + \frac{\tau_1^{l-1}}{(l-1)!} \cdot \frac{\tau_2^{m-1}}{(m-1)!} \cdot \frac{\tau_3^2}{2!} \dots \right]$$

$$\frac{I_{2l+1, 2m+1}}{(2l+1)! (2m+1)!} = (-1)^{l+m+1} D^{1/2} \left[\frac{\tau_1^l}{l!} \cdot \frac{\tau_2^m}{m!} \frac{\tau_3}{1!} + \frac{\tau_1^{l-1}}{(l-1)!} \cdot \frac{\tau_2^{m-1}}{(m-1)!} \frac{\tau_3^3}{3!} \dots \right]$$

$$I_{2l, 2m+1} = I_{2l+1, 2m} = 0;$$

or, ignoring the minus sign and the factor $D^{1/2}$, the same expression as was obtained for the determination of the expressions

$$\int_{-\infty}^{\infty} \int H_{ij}(x, y) \varphi(x, y) e^{\frac{x^2}{N} t_1 + \frac{y^2}{N} t_2 + \frac{xy}{N} t_3}$$

Regrouping finally gives

$$(92) \quad U(t_1, t_2, t_3) = D^{-\frac{N-1}{2}} \left[1 + \sum \sum \left[\sum \frac{C_{ijk}}{i! j! k!} \tau_1^i \tau_2^j \tau_3^k \right] \right]$$

which is thus the characteristic function of the distribution of the central moments of the second order in random samples from a correlation surface of Type A.

The coefficients C_{ijk} are determined by the symbolic relation

$$(93) \quad C_{ijk} = [B_{10000} - B_{00020}]^i [B_{01000} - B_{00002}]^j [B_{00100} - B_{00011}]^k$$

if, after developing, the different products

$$B_{10000}^a B_{01000}^b B_{00100}^c B_{00020}^d B_{00002}^e B_{00011}^f$$

are replaced by

$$B_{abc, 2d+f, 2e+f}$$

For the coefficients up to the fourth order we obtain.

$$(94) \quad \begin{aligned} C_{000} &= 1 \\ C_{100} &= 0 \\ &\dots\dots\dots \\ C_{200} &= \frac{(N-1)^2}{N^3} \lambda_{40} \\ C_{110} &= \frac{(N-1)^2}{N^3} \lambda_{22} + 2 \frac{N-1}{N^2} \lambda_{11}^2 \\ &\dots\dots\dots \end{aligned}$$

$$\begin{aligned}
C_{300} &= \frac{(N-1)^3}{N^5} \lambda_{60} + 4 \frac{(N-1)(N-2)}{N^4} \lambda_{30}^2 \\
C_{210} &= \frac{(N-1)^3}{N^5} \lambda_{42} + 4 \frac{(N-1)(N-2)}{N^4} \lambda_{21}^2 + 8 \frac{(N-1)^2}{N^4} \lambda_{31} \lambda_{11} \\
&\dots\dots\dots \\
C_{400} &= \frac{(N-1)^4}{N^7} \lambda_{80} + 32 \frac{(N-1)^2(N-2)}{N^6} \lambda_{50} \lambda_{30} + \\
&\quad + \frac{(N-1)(3N^3 + 23N^2 - 63N + 45)}{N^6} \lambda_{40}^2 \\
C_{310} &= \frac{(N-1)^4}{N^7} \lambda_{62} + 24 \frac{(N-1)^2(N-2)}{N^6} \lambda_{41} \lambda_{12} + 8 \frac{(N-1)^2(N-2)}{N^6} \lambda_{32} \lambda_{30} + \\
&\quad + 12 \frac{(N-1)^3}{N^6} \lambda_{51} \lambda_{11} + 3 \frac{(N-1)^3(N+3)}{N^6} \lambda_{40} \lambda_{22} + \\
&\quad + 4 \frac{(N-1)(5N^2 - 12N + 9)}{N^6} \lambda_{31}^2 + 6 \frac{(N-1)^2(N+3)}{N^5} \lambda_{40} \lambda_{11}^2 + \\
&\quad + 48 \frac{(N-1)(N-2)}{N^5} \lambda_{30} \lambda_{21} \lambda_{11} \\
(94) \quad C_{220} &= \frac{(N-1)^4}{N^7} \lambda_{44} + 16 \frac{(N-1)^3}{N^6} \lambda_{33} \lambda_{11} + 16 \frac{(N-1)^2(N-2)}{N^6} \lambda_{32} \lambda_{12} + \\
&\quad + 16 \frac{(N-1)^2(N-2)}{N^6} \lambda_{23} \lambda_{21} + \frac{(N-1)^4}{N^6} \lambda_{40} \lambda_{04} + 16 \frac{(N-1)^3}{N^6} \lambda_{31} \lambda_{13} + \\
&\quad + 2 \frac{(N-1)(N^3 + 5N^2 - 17N + 15)}{N^6} \lambda_{22}^2 + 8 \frac{(N-1)^2(N+7)}{N^5} \lambda_{22} \lambda_{11}^2 + \\
&\quad + 64 \frac{(N-1)(N-2)}{N^5} \lambda_{21} \lambda_{12} \lambda_{11} + 8 \frac{(N-1)(N+1)}{N^4} \lambda_{11}^4 \\
&\dots\dots\dots \\
C_{001} &= \frac{N-1}{N} \lambda_{11} \\
C_{101} &= \frac{(N-1)^2}{N^3} \lambda_{31} \\
&\dots\dots\dots \\
C_{201} &= \frac{(N-1)^3}{N^5} \lambda_{51} + \frac{(N-1)^2(N+3)}{N^4} \lambda_{40} \lambda_{11} + 4 \frac{(N-1)(N-2)}{N^4} \lambda_{30} \lambda_{21} \\
C_{111} &= \frac{(N-1)^3}{N^5} \lambda_{33} + \frac{(N-1)^2(N+7)}{N^4} \lambda_{22} \lambda_{11} + 4 \frac{(N-1)(N-2)}{N^4} \lambda_{21} \lambda_{12} + \\
&\quad + 2 \frac{(N-1)(N+1)}{N^3} \lambda_{11}^3 \\
&\dots\dots\dots
\end{aligned}$$

$$\begin{aligned}
C_{301} &= \frac{(N-1)^4}{N^7} \lambda_{71} + \frac{(N-1)^3(N+5)}{N^6} \lambda_{60} \lambda_{11} + 12 \frac{(N-1)^2(N-2)}{N^6} \lambda_{50} \lambda_{21} + \\
&\quad + 20 \frac{(N-1)^2(N-2)}{N^6} \lambda_{41} \lambda_{30} + \frac{(N-1)(3N^3+23N^2-63N+45)}{N^6} \lambda_{40} \lambda_{31} + \\
&\quad + 4 \frac{(N-1)(N-2)(N+5)}{N^5} \lambda_{30}^2 \lambda_{11} \\
C_{211} &= \frac{(N-1)^4}{N^7} \lambda_{53} + \frac{(N-1)^3(N+13)}{N^6} \lambda_{42} \lambda_{11} + 8 \frac{(N-1)^2(N-2)}{N^6} \lambda_{41} \lambda_{12} + \\
&\quad + 20 \frac{(N-1)^2(N-2)}{N^6} \lambda_{32} \lambda_{21} + 4 \frac{(N-1)^2(N-2)}{N^6} \lambda_{23} \lambda_{30} + \\
&\quad + \frac{(N-1)^3(N+3)}{N^6} \lambda_{40} \lambda_{13} + 2 \frac{(N-1)(N^3+11N^2-29N+21)}{N^6} \lambda_{31} \lambda_{22} + \\
&\quad + \frac{(N-1)(N-2)}{N^5} \lambda_{30} \lambda_{12} \lambda_{11} + 4 \frac{(N-1)(N-2)(N+9)}{N^5} \lambda_{21}^2 \lambda_{11} + \\
&\quad + 12 \frac{(N-1)^2(N+3)}{N^5} \lambda_{31} \lambda_{11}^2 \\
&\dots\dots\dots \\
C_{002} &= \frac{(N-1)^2}{N^3} \lambda_{22} + \frac{N-1}{N} \lambda_{11}^2 \\
(94) \quad C_{102} &= \frac{(N-1)^3}{N^5} \lambda_{42} + 2 \frac{(N-1)^2(N+2)}{N^4} \lambda_{31} \lambda_{11} + 2 \frac{(N-1)(N-2)}{N^4} \lambda_{30} \lambda_{12} + \\
&\quad + 2 \frac{(N-1)(N-2)}{N^4} \lambda_{21}^2 \\
&\dots\dots\dots \\
C_{202} &= \frac{(N-1)^4}{N^7} \lambda_{62} + 2 \frac{(N-1)^3(N+4)}{N^6} \lambda_{51} \lambda_{11} + 4 \frac{(N-1)^2(N-2)}{N^6} \lambda_{50} \lambda_{12} + \\
&\quad + 16 \frac{(N-1)^2(N-2)}{N^6} \lambda_{41} \lambda_{21} + 12 \frac{(N-1)^2(N-2)}{N^6} \lambda_{32} \lambda_{30} + \\
&\quad + \frac{(N-1)(N^3+11N^2-29N+21)}{N^6} \lambda_{40} \lambda_{22} + \\
&\quad + 2 \frac{(N-1)(N^3+6N^2-17N+12)}{N^6} \lambda_{31}^2 + \\
&\quad + \frac{(N-1)^2(N+3)(N+4)}{N^5} \lambda_{40} \lambda_{11}^2 + 8 \frac{(N-1)(N-2)(N+4)}{N^5} \lambda_{30} \lambda_{21} \lambda_{11} \\
C_{112} &= \frac{(N-1)^4}{N^7} \lambda_{44} + 2 \frac{(N-1)^3(N+6)}{N^6} \lambda_{33} \lambda_{11} + 2 \frac{(N-1)^2(N-2)}{N^6} \lambda_{41} \lambda_{03} + \\
&\quad + 14 \frac{(N-1)^2(N-2)}{N^6} \lambda_{32} \lambda_{12} + 14 \frac{(N-1)^2(N-2)}{N^6} \lambda_{23} \lambda_{21} + \\
&\quad + 2 \frac{(N-1)^2(N-2)}{N^6} \lambda_{14} \lambda_{30} + \frac{(N-1)^3}{N^6} \lambda_{40} \lambda_{04} +
\end{aligned}$$

$$\begin{aligned}
& + 2 \frac{(N-1)(N^3 + 4N^2 - 13N + 10)}{N^6} \lambda_{31} \lambda_{13} + \\
& + \frac{(N-1)(N^3 + 14N^2 - 35N + 24)}{N^6} \lambda_{22}^2 + \\
& + \frac{(N-1)^2(N^2 + 17N + 34)}{N^5} \lambda_{22} \lambda_{11}^2 + 4 \frac{(N-1)(N-2)}{N^5} \lambda_{30} \lambda_{03} \lambda_{11} + \\
& + 4 \frac{(N-1)(N-2)(2N+11)}{N^5} \lambda_{21} \lambda_{12} \lambda_{11} + \\
& + 2 \frac{(N-1)(N+1)(N+2)}{N^4} \lambda_{11}^4 \\
& \dots\dots\dots \\
C_{003} = & \frac{(N-1)^3}{N^5} \lambda_{33} + \frac{(N-1)(N-2)}{N^4} \lambda_{30} \lambda_{03} + 3 \frac{(N-1)(N-2)}{N^4} \lambda_{21} \lambda_{12} + \\
& + 3 \frac{(N-1)^2(N+1)}{N^4} \lambda_{22} \lambda_{11} + \frac{(N-1)(N+1)}{N^2} \lambda_{11}^3 \\
C_{103} = & \frac{(N-1)^4}{N^7} \lambda_{53} + 3 \frac{(N-1)^3(N+3)}{N^6} \lambda_{42} \lambda_{11} + \frac{(N-1)^2(N-2)}{N^6} \lambda_{50} \lambda_{03} + \\
(94) \quad & + 9 \frac{(N-1)^2(N-2)}{N^6} \lambda_{41} \lambda_{12} + 15 \frac{(N-1)^2(N-2)}{N^6} \lambda_{32} \lambda_{21} + \\
& + 7 \frac{(N-1)^2(N-2)}{N^6} \lambda_{23} \lambda_{30} + \frac{(N-1)(5N^2 - 12N + 9)}{N^6} \lambda_{40} \lambda_{13} + \\
& + 3 \frac{(N-1)(N^3 + 6N^2 - 17N + 12)}{N^6} \lambda_{31} \lambda_{22} + \\
& + 6 \frac{(N-1)(N-2)(N+7)}{N^5} \lambda_{30} \lambda_{12} \lambda_{11} + \\
& + 6 \frac{(N-1)(N-2)(N+3)}{N^5} \lambda_{21}^2 \lambda_{11} + \\
& + 3 \frac{(N-1)^2(N+2)(N+3)}{N^5} \lambda_{31} \lambda_{11}^2 \\
& \dots\dots\dots \\
C_{004} = & \frac{(N-1)^4}{N^7} \lambda_{44} + 4 \frac{(N-1)^3(N+2)}{N^6} \lambda_{33} \lambda_{11} + 4 \frac{(N-1)^2(N-2)}{N^6} \lambda_{41} \lambda_{03} + \\
& + 12 \frac{(N-1)^2(N-2)}{N^6} \lambda_{32} \lambda_{12} + 12 \frac{(N-1)^2(N-2)}{N^6} \lambda_{23} \lambda_{21} + \\
& + 4 \frac{(N-1)^2(N-2)}{N^6} \lambda_{14} \lambda_{30} + \frac{(N-1)(N^2 - 3N + 3)}{N^6} \lambda_{40} \lambda_{04} +
\end{aligned}$$

$$\begin{aligned}
 & + 4 \frac{(N-1)(4N^2-9N+6)}{N^6} \lambda_{31} \lambda_{13} + \\
 & + 3 \frac{(N-1)(N^3+2N^2-8N+6)}{N^6} \lambda_{22}^2 + \\
 & + 6 \frac{(N-1)^2(N+1)(N+2)}{N^5} \lambda_{22} \lambda_{11}^2 + \\
 (94) \quad & + 4 \frac{(N-1)(N-2)(N+2)}{N^5} \lambda_{30} \lambda_{03} \lambda_{11} + \\
 & + 12 \frac{(N-1)(N-2)(N+2)}{N^5} \lambda_{21} \lambda_{12} \lambda_{11} + \\
 & + \frac{(N-1)(N+1)(N+2)}{N^3} \lambda_{11}^4
 \end{aligned}$$

If the characteristic function is written as an exponential expression under the form

$$(95) \quad U(t_1, t_2, t_3) = D^{-\frac{N-1}{2}} e^{\sum \sum \sum \frac{\vartheta_{ijk}}{i!j!k!} \tau_1^i \tau_2^j \tau_3^k},$$

new characteristics are obtained which partially agree with those used in the distribution of the second moment and from which it is easy to pass over to the determining of the seminvariants.

The following expressions are obtained for these characteristics up to the fourth order.

$$\begin{aligned}
 \vartheta_{100} &= 0 \\
 &\dots\dots\dots \\
 \vartheta_{200} &= \frac{(N-1)^2}{N^3} \lambda_{40} \\
 \vartheta_{110} &= \frac{(N-1)^2}{N^3} \lambda_{22} + 2 \frac{(N-1)}{N^2} \lambda_{11}^2 \\
 &\dots\dots\dots \\
 (96) \quad \vartheta_{300} &= \frac{(N-1)^3}{N^5} \lambda_{60} + 4 \frac{(N-1)(N-2)}{N^4} \lambda_{30}^2 \\
 \vartheta_{210} &= \frac{(N-1)^3}{N^5} \lambda_{42} + 8 \frac{(N-1)^2}{N^4} \lambda_{31} \lambda_{11} + 4 \frac{(N-1)(N-2)}{N^4} \lambda_{21}^2 \\
 &\dots\dots\dots \\
 \vartheta_{400} &= \frac{(N-1)^4}{N^7} \lambda_{80} + 32 \frac{(N-1)^2(N-2)}{N^6} \lambda_{50} \lambda_{30} + 8 \frac{(N-1)(4N^2-9N+6)}{N^6} \lambda_{40}^2
 \end{aligned}$$

$$\begin{aligned}
\vartheta_{310} &= \frac{(N-1)^4}{N^7} \lambda_{62} + 12 \frac{(N-1)^3}{N^6} \lambda_{51} \lambda_{11} + 24 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{41} \lambda_{21} + \\
&+ 8 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{32} \lambda_{30} + 12 \frac{(N-1)^3}{N^6} \lambda_{40} \lambda_{22} + \\
&+ 4 \frac{(N-1) (5N^2 - 12N + 9)}{N^6} \lambda_{31}^2 + 24 \frac{(N-1)^2}{N^5} \lambda_{40} \lambda_{11}^2 + \\
&+ 48 \frac{(N-1) (N-2)}{N^5} \lambda_{30} \lambda_{21} \lambda_{11} \\
\vartheta_{220} &= \frac{(N-1)^4}{N^7} \lambda_{44} + 16 \frac{(N-1)^3}{N^6} \lambda_{33} \lambda_{11} + 16 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{32} \lambda_{12} + \\
&+ 16 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{23} \lambda_{21} + 16 \frac{(N-1)^3}{N^6} \lambda_{31} \lambda_{13} + \\
&+ 8 \frac{(N-1) (2N^2 - 5N + 4)}{N^6} \lambda_{22}^2 + 64 \frac{(N-1)^2}{N^5} \lambda_{22} \lambda_{11}^2 + \\
&- 64 \frac{(N-1) (N-2)}{N^5} \lambda_{21} \lambda_{12} \lambda_{11} + 16 \frac{N-1}{N^4} \lambda_{11}^4
\end{aligned}$$

.....

$$(96) \quad \vartheta_{001} = \frac{N-1}{N} \lambda_{11}$$

$$\vartheta_{101} = \frac{(N-1)^2}{N^3} \lambda_{31}$$

.....

$$\vartheta_{201} = \frac{(N-1)^3}{N^5} \lambda_{51} + 4 \frac{(N-1)^2}{N^4} \lambda_{40} \lambda_{11} + 4 \frac{(N-1) (N-2)}{N^4} \lambda_{30} \lambda_{21}$$

$$\begin{aligned}
\vartheta_{111} &= \frac{(N-1)^3}{N^5} \lambda_{33} + 8 \frac{(N-1)^2}{N^4} \lambda_{22} \lambda_{11} + 4 \frac{(N-1) (N-2)}{N^4} \lambda_{21} \lambda_{12} + \\
&+ 4 \frac{(N-1)}{N^3} \lambda_{11}^3
\end{aligned}$$

.....

$$\begin{aligned}
\vartheta_{301} &= \frac{(N-1)^4}{N^7} \lambda_{71} + 6 \frac{(N-1)^3}{N^6} \lambda_{60} \lambda_{11} + 12 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{50} \lambda_{21} + \\
&+ 20 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{41} \lambda_{30} + 8 \frac{(N-1) (4N^2 - 9 + 6)}{N^6} \lambda_{40} \lambda_{31} + \\
&+ 24 \frac{(N-1) (N-2)}{N^5} \lambda_{30}^2 \lambda_{11}
\end{aligned}$$

$$\vartheta_{211} = \frac{(N-1)^4}{N^7} \lambda_{53} + 14 \frac{(N-1)^3}{N^6} \lambda_{42} \lambda_{11} + 8 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{41} \lambda_{12} +$$

$$\begin{aligned}
& + 20 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{32} \lambda_{21} + 4 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{23} \lambda_{30} + \\
& + 4 \frac{(N-1)^3}{N^6} \lambda_{40} \lambda_{13} + 2 \frac{(N-1)(14N^2 - 32N + 22)}{N^6} \lambda_{31} \lambda_{22} + \\
& + \frac{(N-1)(N-2)}{N^5} \lambda_{30} \lambda_{12} \lambda_{11} + 40 \frac{(N-1)(N-2)}{N^5} \lambda_{21}^2 \lambda_{11} + \\
& + 48 \frac{(N-1)^2}{N^5} \lambda_{31} \lambda_{11}^2 \\
& \dots\dots\dots \\
\vartheta_{002} &= \frac{(N-1)^2}{N^3} \lambda_{22} + \frac{N-1}{N^2} \lambda_{11}^2 \\
\vartheta_{102} &= \frac{(N-1)^3}{N^5} \lambda_{42} + 6 \frac{(N-1)^2}{N^4} \lambda_{31} \lambda_{11} + 2 \frac{(N-1)(N-2)}{N^4} \lambda_{30} \lambda_{12} + \\
& + 2 \frac{(N-1)(N-2)}{N^4} \lambda_{21}^2 \\
& \dots\dots\dots \\
\vartheta_{202} &+ \frac{(N-1)^4}{N^7} \lambda_{62} + 10 \frac{(N-1)^3}{N^6} \lambda_{51} \lambda_{11} + 4 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{50} \lambda_{12} + \\
(96) \quad & + 16 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{41} \lambda_{21} + 12 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{32} \lambda_{30} + \\
& + \frac{(N-1)(14N^2 - 32N + 22)}{N^6} \lambda_{40} \lambda_{22} + \\
& + 2 \frac{(N-1)(9N^2 - 20N + 13)}{N^6} \lambda_{31}^2 + \\
& + 20 \frac{(N-1)^2}{N^5} \lambda_{40} \lambda_{11}^2 + 40 \frac{(N-1)(N-2)}{N^5} \lambda_{30} \lambda_{21} \lambda_{11} \\
\vartheta_{112} &= \frac{(N-1)^4}{N^7} \lambda_{44} + 14 \frac{(N-1)^3}{N^6} \lambda_{33} \lambda_{11} + 2 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{41} \lambda_{03} + \\
& + 14 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{32} \lambda_{13} + 14 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{23} \lambda_{21} + \\
& + 2 \frac{(N-1)^2 (N-2)}{N^6} \lambda_{14} \lambda_{30} + \frac{(N-1)^3}{N^6} \lambda_{40} \lambda_{04} + \\
& + 2 \frac{(N-1)(7N^2 - 16N + 11)}{N^6} \lambda_{31} \lambda_{13} + \frac{(N-1)(17N^2 - 38N + 25)}{N^6} \lambda_{22}^2 + \\
& + 52 \frac{(N-1)^2}{N^5} \lambda_{22} \lambda_{11}^2 + 4 \frac{(N-1)(N-2)}{N^5} \lambda_{30} \lambda_{03} \lambda_{11} + \\
& + 52 \frac{(N-1)(N-2)}{N^5} \lambda_{21} \lambda_{12} \lambda_{11} + 12 \frac{(N-1)}{N^4} \lambda_{11}^4 \\
& \dots\dots\dots
\end{aligned}$$

$$\begin{aligned}
 \vartheta_{003} &= \frac{(N-1)^3}{N^5} \lambda_{33} + \frac{(N-1)(N-2)}{N^4} \lambda_{30} \lambda_{03} + 3 \frac{(N-1)(N-2)}{N^4} \lambda_{21} \lambda_{12} + \\
 &\quad + 6 \frac{(N-1)^2}{N^4} \lambda_{22} \lambda_{11} + 2 \frac{(N-1)}{N^3} \lambda_{11}^3 \\
 \vartheta_{103} &= \frac{(N-1)^4}{N^7} \lambda_{53} + 12 \frac{(N-1)^3}{N^6} \lambda_{42} \lambda_{11} + \frac{(N-1)^2(N-2)}{N^6} \lambda_{50} \lambda_{03} + \\
 &\quad + 9 \frac{(N-1)^2(N-2)}{N^6} \lambda_{41} \lambda_{12} + 15 \frac{(N-1)^2(N-2)}{N^6} \lambda_{32} \lambda_{21} + \\
 &\quad + 7 \frac{(N-1)^2(N-2)}{N^6} \lambda_{23} \lambda_{30} + \frac{(N-1)(5N^2-12N+9)}{N^6} \lambda_{40} \lambda_{13} + \\
 &\quad + 3 \frac{(N-1)(9N^2-20N+13)}{N^6} \lambda_{31} \lambda_{22} + 48 \frac{(N-1)(N-2)}{N^5} \lambda_{30} \lambda_{12} \lambda_{11} + \\
 &\quad + 24 \frac{(N-1)(N-2)}{N^5} \lambda_{21}^2 \lambda_{11} + 36 \frac{(N-1)^2}{N^5} \lambda_{31} \lambda_{11}^2 \\
 &\quad \dots\dots\dots \\
 \vartheta_{004} &+ \frac{(N-1)^4}{N^7} \lambda_{44} + 12 \frac{(N-1)^3}{N^6} \lambda_{33} \lambda_{11} + 4 \frac{(N-1)^2(N-2)}{N^6} \lambda_{41} \lambda_{03} + \\
 &\quad + 12 \frac{(N-1)^2(N-2)}{N^6} \lambda_{32} \lambda_{12} + 12 \frac{(N-1)^2(N-2)}{N^6} \lambda_{23} \lambda_{21} + \\
 &\quad + 4 \frac{(N-1)^2(N-2)}{N^6} \lambda_{14} \lambda_{30} + \frac{(N-1)(N^2-3N+3)}{N^6} \lambda_{40} \lambda_{04} + \\
 &\quad + 4 \frac{(N-1)(4N^2-9N+6)}{N^6} \lambda_{31} \lambda_{13} + 3 \frac{(N-1)(5N^2-11N+7)}{N^6} \lambda_{22}^2 + \\
 &\quad + 36 \frac{(N-1)^2}{N^5} \lambda_{22} \lambda_{11}^2 + 12 \frac{(N-1)(N-2)}{N^5} \lambda_{30} \lambda_{03} \lambda_{11} + \\
 &\quad + 36 \frac{(N-1)(N-2)}{N^5} \lambda_{21} \lambda_{12} \lambda_{11} + 6 \frac{(N-1)}{N^4} \lambda_{11}^4
 \end{aligned}
 \tag{97}$$

In the event of the two variates in the parent population being completely independent ($\lambda_{ij} = 0$ if both i and $j \neq 0$) the following expressions are obtained.

$$\begin{aligned}
 \vartheta_{100} &= 0 \\
 \vartheta_{010} &= 0 \\
 &\dots\dots\dots \\
 \vartheta_{200} &= \frac{(N-1)^2}{N^3} \lambda_{40} \\
 \vartheta_{110} &= 0 \\
 &\dots\dots\dots
 \end{aligned}$$

$$\begin{aligned}
\vartheta_{300} &= \frac{(N-1)^3}{N^5} \lambda_{60} + 4 \frac{(N-1)(N-2)}{N^4} \lambda_{30}^2 \\
\vartheta_{210} &= \vartheta_{120} = 0 \\
&\dots\dots\dots \\
\vartheta_{400} &= \frac{(N-1)^4}{N^7} \lambda_{80} + 32 \frac{(N-1)^2(N-2)}{N^6} \lambda_{50} \lambda_{30} + 8 \frac{(N-1)(4N^2-9N+6)}{N^6} \lambda_{40}^2 \\
\vartheta_{310} &= \vartheta_{220} = \vartheta_{130} = 0 \\
&\dots\dots\dots \\
\vartheta_{001} &= 0 \\
\vartheta_{101} &= 0 \\
\vartheta_{011} &= 0 \\
\vartheta_{201} &= 0 \\
\vartheta_{111} &= 0 \\
\vartheta_{021} &= 0 \\
\vartheta_{301} &= 0 \\
\vartheta_{211} &= 0 \\
\vartheta_{121} &= 0 \\
\vartheta_{031} &= 0 \\
(97) \quad &\dots\dots\dots \\
&\vartheta_{002} = 0 \\
&\vartheta_{102} = 0 \\
&\vartheta_{012} = 0 \\
&\vartheta_{202} = 0 \\
&\vartheta_{112} = \frac{(N-1)^3}{N^6} \lambda_{40} \lambda_{04} \\
&\vartheta_{022} = 0 \\
&\dots\dots\dots \\
&\vartheta_{003} = \frac{(N-1)(N-2)}{N^4} \lambda_{30} \lambda_{03} \\
&\vartheta_{103} = \frac{(N-1)^2(N-2)}{N^6} \lambda_{50} \lambda_{03} \\
&\vartheta_{013} = \frac{(N-1)^2(N-2)}{N^6} \lambda_{05} \lambda_{30} \\
&\dots\dots\dots \\
&\vartheta_{004} = \frac{(N-1)(N^2-3N+3)}{N^6} \lambda_{40} \lambda_{04} \\
&\dots\dots\dots
\end{aligned}$$

I will not pass over to the corresponding formulas for the seminvariants. The seminvariants will not be of any use in this type of multivariate distributions and the characteristics given above will be of greater use when we study the properties of such multivariate distributions.

The magnitude of the different coefficients depends both on the magnitude of the seminvariants contained in them and on the number of individuals in the sample.

Applying the first formula for the bivariate correlation distribution, that is to say, λ_{11} is not retained in the expressions for the A-coefficients, we find that the C-coefficients are of the following order in relation to N .

$$C_{ijk} \sim N^{-\frac{i+j+k}{2}} \text{ if } i+j+k \text{ is even}$$

$$C_{ijk} \sim N^{-\frac{i+j+k+1}{2}} \text{ if } i+j+k \text{ is odd.}$$

In relation to s the coefficients are of the following order

$$C_{ijk} \sim C_{i+j+k,00}$$

As stated before (Chapter 8)

$$C_{3n-2,00} \sim C_{3n-1,00} \sim C_{3n,00} \sim s^{-n}$$

On the other hand, if the bivariate distribution has been written so that λ_{11} is retained in the expressions for the A-coefficients, the magnitude of the different coefficients will be changed in so far that the magnitude of the coefficients of C_{ij0} will be unaltered but the magnitude of the coefficients of type C_{ijk} will be the same as that of C_{ij0} . This applies both in relation to N and to s . If, however, λ_{11} or rather r is a small quantity,

$$C_{ijk} \sim r^k C_{ij0}$$

that is to say, in such a case all coefficients will decrease rather rapidly with increasing order, although all coefficients will not decrease with growing N .

CHAPTER 11.

The Distribution of the Moments of the Second Order.

Now that the characteristic function

$$U(t_1, t_2, t_3) = D^{-\frac{N-1}{2}} \left[1 + \sum \sum \sum \frac{C_{lmn}}{l! m! n!} \tau_1^l \tau_2^m \tau_3^n \right]$$

of the correlation distribution of the three central moments of the second order is thus determined, I shall proceed to deduce the corresponding law of distribution.

$$(98) \quad F(\nu_{20}, \nu_{02}, \nu_{11}) = \left(\frac{1}{2\pi} \right)^3 \int \int \int_{-\infty}^{\infty} U e^{-\nu_{20}\omega_1 i - \nu_{02}\omega_2 i - \nu_{11}\omega_3 i} d\omega_1 d\omega_2 d\omega_3.$$

The formula for this distribution must contain a principal term, identical with the expression that represents the distribution in the special case of the parent population being a bivariate normal correlation distribution, and a triple series of terms containing derivatives of expressions almost identical with the principal term (the exponent differs by having its power raised as many steps as the sum of the exponents for τ_1, τ_2 , and τ_3 indicates.)

The formula of the principal term is as before

$$(99) \quad F_0 = C_0 [\nu_{20} \nu_{02} - \nu_{11}^2]^{\frac{N-4}{2}} e^{-\frac{N}{2(1-r^2)} [\nu_{20} + \nu_{02} - 2r\nu_{11}]}.$$

The expression $D^{-\frac{N-1}{2} - n}$ corresponds to a distribution of the type

$$(100) \quad F_n = C_n [\nu_{20} \nu_{02} - \nu_{11}^2]^{\frac{N-4}{2} + n} e^{-\frac{N}{2(1-r^2)} [\nu_{20} + \nu_{02} - 2r\nu_{11}]},$$

where

$$(101) \quad C_n = C_0 \cdot \left(\frac{N}{1-r^2} \right)^{2n} \frac{(N-3)!}{(N+2n-3)!}.$$

The expressions

$$\tau_1 = \frac{t_1 - \frac{1}{N} \left[2t_1 t_2 - \frac{1}{2} t_3^2 \right]}{\left[1 - \frac{2t_1 + 2t_2 + 2r t_3}{N} + \frac{4t_1 t_2 - t_3^2}{N^2} (1 - r^2) \right]};$$

$$\tau_3 = \frac{t_3 + \frac{r}{N} [4t_1 t_2 - t_3^2]}{\left[1 - \frac{2t_1 + 2t_2 + 2r t_3}{N} + \frac{4t_1 t_2 - t_3^2}{N^2} (1 - r^2) \right]};$$

can be written

$$\tau_1 = \frac{N}{2(1 - r^2)} \left[\frac{1 - \frac{2r^2 t_1 + 2t_2 + 2r t_3}{N}}{D} - 1 \right];$$

$$\tau_3 = \frac{N}{(1 - r^2)} \left[r - \frac{r - \frac{2r t_1 + 2r t_2 + t_3(1 + r^2)}{N}}{D} \right].$$

Since the expression $t_1 \cdot U$ corresponds to the derivative

$$-\frac{\partial}{\partial v_{20}} f(v_{20}, v_{02}, v_{11}),$$

we find that the term

$$\tau_1^l \tau_2^m \tau_3^n U_0(t_1, t_2, t_3)$$

in the characteristic function corresponds to an expression f_{lmn} in the frequency function, which can be symbolically written

$$\begin{aligned} f_{lmn} = & \left(\frac{1}{2} \right)^{l+m} \left(\frac{N}{1 - r^2} \right)^{l+m+n} \left[F_1 + \frac{2}{N} \left(r^2 \frac{\partial}{\partial v_{20}} + \frac{\partial}{\partial v_{02}} + r \frac{\partial}{\partial v_{11}} \right) F_1 - 1 \right]^l \\ (102) \quad & \left[F_1 + \frac{2}{N} \left(\frac{\partial}{\partial v_{20}} + r^2 \frac{\partial}{\partial v_{02}} + r \frac{\partial}{\partial v_{11}} \right) F_1 - 1 \right]^m \left[r - r F_1 - \frac{1}{N} \left(2r \frac{\partial}{\partial v_{20}} + \right. \right. \\ & \left. \left. + 2r \frac{\partial}{\partial v_{02}} + (1 + r^2) \frac{\partial}{\partial v_{11}} \right) F_1 \right]^n F_0 \end{aligned}$$

if, when this expression is expanded into a series according to powers of F_1 , the expression

$$F_1^a F_1^b F_1^c F_0$$

is replaced by F_{a+b+c} and the expression

$$\left(\frac{\delta}{\delta v_{20}}\right)^a \left(\frac{\delta}{\delta v_{02}}\right)^b \left(\frac{\delta}{\delta v_{11}}\right)^c F_{a+b+c}$$

is replaced by

$$\frac{\delta^{a+b+c}}{\delta v_{20}^a \delta v_{02}^b \delta v_{11}^c} F_{a+b+c}.$$

If F_n is written

(103 a)

$$e^{-\frac{N}{2(1-r^2)}[v_{20}+v_{02}-2r v_{11}]} \cdot \psi_n,$$

where

(103 b)

$$\psi_n = C_n \cdot [v_{20} v_{02} - v_{11}^2]^{\frac{N-4}{2}+n},$$

we find that

$$\begin{aligned} F_1 + \frac{2}{N} \left(r^2 \frac{\delta}{\delta v_{20}} + \frac{\delta}{\delta v_{02}} + r \frac{\delta}{\delta v_{11}} \right) F_1 = \\ = e^{-\frac{N}{2(1-r^2)}[v_{20}+v_{02}-2r v_{11}]} \cdot \frac{2}{N} \left[r^2 \frac{\delta}{\delta v_{20}} + \frac{\delta}{\delta v_{02}} + r \frac{\delta}{\delta v_{11}} \right] \psi_1 \end{aligned}$$

and

$$\begin{aligned} -r F_1 - \frac{1}{N} \left[2r \frac{\delta}{\delta v_{20}} + 2r \frac{\delta}{\delta v_{02}} + (1+r^2) \frac{\delta}{\delta v_{11}} \right] F_1 = \\ = e^{-\frac{N}{2(1-r^2)}[v_{20}+v_{02}-2r v_{11}]} \cdot \frac{1}{N} \left[2r \frac{\delta}{\delta v_{20}} + 2r \frac{\delta}{\delta v_{02}} + (1+r^2) \frac{\delta}{\delta v_{11}} \right] \psi_1. \end{aligned}$$

Hence f_{lmn} symbolically can be written

$$\begin{aligned} f_{lmn} = e^{-\frac{N}{2(1-r^2)}[v_{20}+v_{02}-2r v_{11}]} \cdot \\ \cdot \left(\frac{1}{2}\right)^{l+m} \left(\frac{N}{1-r^2}\right)^{l+m+n} \left[\frac{2}{N} \left(r^2 \frac{\delta}{\delta v_{20}} + \frac{\delta}{\delta v_{02}} + r \frac{\delta}{\delta v_{11}} \right) \psi_1 - 1 \right]^l \cdot \\ (104) \quad \cdot \left[\frac{2}{N} \left(\frac{\delta}{\delta v_{20}} + r^2 \frac{\delta}{\delta v_{02}} + r \frac{\delta}{\delta v_{11}} \right) \psi_1 - 1 \right]^m \left[r - \frac{1}{N} \left(2r \frac{\delta}{\delta v_{20}} + 2r \frac{\delta}{\delta v_{02}} + \right. \right. \\ \left. \left. + (1+r^2) \frac{\delta}{\delta v_{11}} \right) \psi_1 \right]^n \psi_0. \end{aligned}$$

The general case $r \neq 0$ will not be further considered, but only the special case $r=0$. This, however, as previously mentioned, does not necessarily mean that the mixed second moment of the parent population is identically equal to zero, but that the parent population is described by a »Charlier's Correlation Surface of Type A», where r is certainly equal to zero but where the coefficient A_{11} is included. This special formula cannot, however, be used in practice unless the correlation coefficient is small. (How large the correlation coefficient may be, will be stated later.)

If $r = 0$, then

$$(105) \quad f_{lmn} = e^{-\frac{N}{2}(\nu_{20} + \nu_{02})} \cdot \left[\frac{\delta}{\delta \nu_{02}} \psi_1 - \frac{N}{2} \right]^l \left[\frac{\delta}{\delta \nu_{20}} \psi_1 - \frac{N}{2} \right]^m \left[-\frac{\delta}{\delta \nu_{11}} \psi_1 \right]^n \psi_0,$$

where

$$\psi_0 = C_0 [\nu_{20} \nu_{02} - \nu_{11}^2]^{\frac{N-4}{2}}$$

$$\psi_n = C_n [\nu_{20} \nu_{02} - \nu_{11}^2]^{\frac{N-4}{2} + n}; \quad C_n = C_0 \cdot \frac{(N-3)!}{(N+2n-3)!};$$

Case a) $l \neq 0; m = 0; n = 0;$

$$f_{l00} = e^{-\frac{N}{2}(\nu_{20} + \nu_{02})} \left[\frac{\delta^l \psi_l}{\delta \nu_{02}^l} - l \frac{N}{2} \frac{\delta^{l-1} \psi_{l-1}}{\delta \nu_{02}^{l-1}} + \dots (-1)^l \left(\frac{N}{2} \right)^l \psi_0 \right]$$

but

$$\frac{\delta^l \psi_l}{\delta \nu_{02}^l} = \psi_0 \left(\frac{N}{2} \right)^l \frac{N^l}{(N+2l-3)(N+2l-5) \dots (N-1)} \nu_{20}^l.$$

The expression inside the parenthesis will be recognized as the polynomial defined in Chapter 3 formulas (24) and (25) by the following relation

$$f_{\frac{N-1}{2}}(\nu_{20}) \frac{N-1}{2} L_l(\nu_{20}) = (-1)^l f_{\frac{N-1}{2}+l}^{(l)}(\nu_{20}).$$

Hence

$$(106 \text{ a}) \quad f_{l00} = F_0(\nu_{20}, \nu_{02}, \nu_{11}) \cdot \frac{N-1}{2} L_l(\nu_{20}).$$

Case b) $l = 0; m = 0; n \neq 0;$

$$(106 \text{ b}) \quad f_{00n} = e^{-\frac{N}{2}(\nu_{20} + \nu_{02})} (-1)^n \frac{\delta^n \psi_n}{\delta \nu_{11}^n}.$$

The different derivatives $\frac{\delta^n \psi_n}{\delta \nu_{11}^n}$ can be expressed as a product by ψ_0 and a polynomial of degree n .

For the first four derivatives we have the following expressions

$$(107 \text{ a}) \quad \begin{aligned} -\frac{\delta \psi_1}{\delta \nu_{11}} &= \psi_0 \cdot \frac{N^2}{(N-1)(N-2)} (N-2) \nu_{11} \\ \frac{\delta^2 \psi_2}{\delta \nu_{11}^2} &= \psi_0 \frac{N^4}{(N+1)N(N-1)(N-2)} [N(N-1)\nu_{11}^2 - N\nu_{20}\nu_{02}] \\ -\frac{\delta^3 \psi_3}{\delta \nu_{11}^3} &= \psi_0 \frac{N^6}{(N+3) \dots (N-2)} [N+2)(N+1)N\nu_{11}^3 - \\ &\quad - 3(N+2)N\nu_{11}\nu_{20}\nu_{02}] \\ \frac{\delta^4 \psi_4}{\delta \nu_{11}^4} &= \psi_0 \frac{N^8}{(N+5) \dots (N-2)} [(N+4)(N+3)(N+2)(N+1)\nu_{11}^4 - \\ &\quad - 6(N+4)(N+2)(N+1)\nu_{11}^2\nu_{20}\nu_{02} + 3(N+4)(N+2)\nu_{20}^2\nu_{02}^2]. \end{aligned}$$

The even derivatives may also be expressed in the following general ways

$$\begin{aligned}
 \frac{\delta^{2n} \psi_{2n}}{\delta v_{11}^{2n}} &= N^{2n} \left[\frac{(N^2 v_{20} v_{02})^n \psi_0}{(N+4n-3) \dots (N+1)(N-1)} - \right. \\
 &- n \frac{(N^2 v_{20} v_{02})^{n-1} \psi_1 (N+2n-3)}{(N+4n-3) \dots (N+3)(N+1)} + \\
 (107 \text{ b}) \quad &\left. \binom{n}{2} \frac{(N^2 v_{20} v_{02})^{n-2} \psi_2 (N+2n-1)(N+2n-3)}{(N+4n-3) \dots (N+5)(N+3)} - \dots + \right. \\
 &\left. + (-1)^n \psi_n \frac{(N+4n-5)(N+4n-7) \dots (N+2n-3)}{(N+4n-3)(N+4n-5) \dots (N+2n-1)} \right].
 \end{aligned}$$

Case c) $l \neq 0$; $m \neq 0$; $n = 0$;

$$f_{lm0} = e^{-\frac{N}{2}(v_{20} + v_{02})} \left[\frac{\delta}{\delta v_{02}} \psi_1 - \frac{N}{2} \right]^l \left[\frac{\delta}{\delta v_{20}} \psi_1 - \frac{N}{2} \right]^m \psi_0.$$

For $m = 1$ we get

$$\begin{aligned}
 f_{l10} &= \frac{\delta}{\delta v_{20}} \left[\psi_1 \cdot \frac{N-1}{2} + 1 L_l(v_{20}) \right] - \frac{N}{2} \psi_0 \frac{N-1}{2} L_l(v_{20}) = \\
 &= \psi_0 \left[\frac{N}{2} \frac{N v_{02}}{N-1} \frac{N-1}{2} + 1 L_l(v_{20}) - \frac{N}{2} \frac{N-1}{2} L_l(v_{20}) \right] + \psi_1 \cdot \frac{\delta}{\delta v_{20}} \frac{N-1}{2} + 1 L_l(v_{20}).
 \end{aligned}$$

but

$$\frac{N-1}{2} + 1 L_l(v_{20}) = \frac{N-1}{2} L_l(v_{20}) - 2l \frac{N^2 v_{20}}{2(N-1)(N+1)} \frac{N-1}{2} + 2 L_{l-1}(v_{20})$$

and

$$\frac{\delta}{\delta v_{20}} \frac{N-1}{2} + 1 L_l(v_{20}) = l \cdot \frac{N^2}{2(N+1)} \frac{N-1}{2} + 2 L_{l-1}(v_{20})$$

and hence we obtain

$$f_{l10} = \psi_0 \cdot \frac{N-1}{2} L_l(v_{20}) \cdot \frac{N-1}{2} L_l(v_{02}) - \frac{l}{2} \frac{N-1}{2} + 2 L_{l-1}(v_{20}) \cdot$$

$$\left[\frac{N^4 v_{20} v_{02}}{(N-1)^2 (N+1)} \psi_0 - \frac{N^2}{(N+1)} \psi_1 \right]$$

or

$$(106 \text{ c}) \quad f_{l10} = \psi_0 \frac{N-1}{2} L_l(v_{20}) \frac{N-1}{2} L_l(v_{02}) - \frac{l}{2! (N-1)} \frac{N-1}{2} + 2 L_{l-1}(v_{20}) \cdot \frac{\delta^2 \psi_2}{\delta v_{11}^2}.$$

For the general case we get in the same manner

$$\begin{aligned}
 f_{lm0} = & e^{-\frac{N}{2}(\nu_{20} + \nu_{02})} \left[\frac{N-1}{2} L_l(\nu_{20}) \frac{N-1}{2} L_m(\nu_{02}) \psi_0 - \right. \\
 & - \frac{1}{2} l \cdot m \frac{N-1}{2} + 2 L_{l-1}(\nu_{20}) \frac{N-1}{2} + 2 L_{m-1}(\nu_{02}) \frac{1}{N-1} \frac{\delta^2 \psi_2}{\delta \nu_{11}^2} + \dots \\
 & + (-1)^i \left(\frac{1}{2} \right)^i i! \binom{l}{i} \binom{m}{i} \frac{N-1}{2} + 2i L_{l-i}(\nu_{20}) \frac{N-1}{2} + 2i L_{m-i}(\nu_{02}) \cdot \\
 & \cdot \frac{1}{(N+2i-3) \dots (N+4i-5)} \frac{\delta^{2i} \psi_{2i}}{\delta \nu_{11}^{2i}} + \dots \\
 & + (-1)^m \binom{l}{m} m! \binom{l}{m} \frac{N-1}{2} + 2m L_{l-m}(\nu_{20}) \cdot \\
 & \cdot \left. \frac{1}{(N+2m-3)(N+2m-1) \dots (N+4m-5)} \frac{\delta^{2m} \psi_{2m}}{\delta \nu_{11}^{2m}} \right].
 \end{aligned}
 \tag{106 d}$$

Case d)

As a consequence of *case c* we immediately obtain

$$\begin{aligned}
 f_{lmk} = & (-1)^k e^{-\frac{N}{2}(\nu_{20} + \nu_{02})} \left[\frac{N-1}{2} + k L_l(\nu_{20}) \frac{N-1}{2} + k L_m(\nu_{02}) \frac{\delta^k}{\delta \nu_{11}^k} \psi_k - \right. \\
 & - \frac{1}{2} \binom{l}{1} \binom{m}{1} \frac{N-1}{2} + k + 2 L_{l-1}(\nu_{20}) \frac{N-1}{2} + k + 2 L_{m-1}(\nu_{02}) \frac{1}{N+2k-1} \frac{\delta^{k+2} \psi_{k+2}}{\delta \nu_{11}^{k+2}} + \dots \left. \right].
 \end{aligned}
 \tag{106 e}$$

Hence we find that $F(\nu_{20}, \nu_{02}, \nu_{11})$ can be written in the following form

$$\begin{aligned}
 F(\nu_{20}, \nu_{02}, \nu_{11}) = & F_0(\nu_{20}, \nu_{02}, \nu_{11}) + \left[\sum \sum \frac{D_{ij0}}{i!j!} \frac{N-1}{2} L_i(\nu_{20}) \frac{N-1}{2} L_j(\nu_{02}) \right] F_0 - \\
 & - \left[\sum \sum \frac{D_{ij1}}{i!j!} \frac{N-1}{2} + 1 L_i(\nu_{20}) \frac{N-1}{2} + 1 L_j(\nu_{02}) \right] \frac{\delta F_1}{\delta \nu_{11}} + \\
 & + \frac{1}{2!} \left[\sum \sum \frac{D_{ij2}}{i!j!} \frac{N-1}{2} + 2 L_i(\nu_{20}) \frac{N-1}{2} + 2 L_j(\nu_{02}) \right] \frac{\delta^2 F_2}{\delta \nu_{11}^2} - \dots
 \end{aligned}
 \tag{108}$$

where

$$F_n = \frac{N^{N+2n-1}}{4\pi(N+2n-3)!} [\nu_{20} \nu_{02} - \nu_{11}^2]^{\frac{N-4}{2} + n} e^{-\frac{N}{2}(\nu_{20} + \nu_{02})},$$

and

$$\begin{aligned}
 D_{ij0} &= C_{ij0} \\
 D_{ij1} &= C_{ij1} \\
 D_{ij2} &= C_{ij2} - \frac{C_{i+1,j+1,0}}{N-1} \\
 (109) \quad D_{ij3} &= C_{ij3} - 3 \frac{C_{i+1,j+1,1}}{N+1} \\
 D_{ij4} &= C_{ij4} - 6 \frac{C_{i+1,j+1,2}}{N+3} + 3 \frac{C_{i+2,j+2,0}}{(N+1)(N+3)} \\
 &\dots\dots\dots
 \end{aligned}$$

We obtain

$$\begin{aligned}
 D_{002} &= \frac{(N-1)(N-2)}{N^3} \lambda_{22} + \frac{(N+1)(N-2)}{N^2} \lambda_{11}^2; \\
 D_{102} &= \frac{(N-1)^2(N-2)}{N^5} \lambda_{42} + 2 \frac{(N-1)(N-2)(N+3)}{N^4} \lambda_{31} \lambda_{11} + \\
 &\quad + 2 \frac{(N-1)(N-2)}{N^4} \lambda_{30} \lambda_{12} + 2 \frac{(N-2)(N-3)}{N^4} \lambda_{21}^2; \\
 &\dots\dots\dots \\
 D_{202} &= \frac{(N-1)^3(N-2)}{N^7} \lambda_{62} + 2 \frac{(N-1)^2(N-2)(N+5)}{N^6} \lambda_{51} \lambda_{11} + \\
 &\quad + 4 \frac{(N-1)^2(N-2)}{N^6} \lambda_{50} \lambda_{12} + 8 \frac{(N-1)(N-2)(2N-5)}{N^6} \lambda_{41} \lambda_{21} + \\
 (110) \quad &\quad + 4 \frac{(N-1)(N-2)(3N-5)}{N^6} \lambda_{32} \lambda_{30} + \\
 &\quad + \frac{(N-1)(N-2)[N^2+10N-15]}{N^6} \lambda_{40} \lambda_{22} + \\
 &\quad + 2 \frac{(N-2)[N^3+7N^2-19N+15]}{N^6} \lambda_{31}^2 + \\
 &\quad + \frac{(N-1)(N-2)(N+3)(N+5)}{N^5} \lambda_{40} \lambda_{11}^2 + 8 \frac{(N-2)^2(N+5)}{N^5} \lambda_{30} \lambda_{21} \lambda_{11} \\
 D_{112} &= \frac{(N-1)^3(N-2)}{N^7} \lambda_{44} + 2 \frac{(N-1)^2(N-2)(N+7)}{N^6} \lambda_{33} \lambda_{11} + \\
 &\quad + 2 \frac{(N-1)^2(N-2)}{N^6} (\lambda_{41} \lambda_{03} + \lambda_{14} \lambda_{30}) + \\
 &\quad + 2 \frac{(N-1)(N-2)(7N-15)}{N^6} (\lambda_{32} \lambda_{12} + \lambda_{23} \lambda_{21}) +
 \end{aligned}$$

$$\begin{aligned}
& + 2 \frac{(N-1)(N-2)(N^2+6N-9)}{N^6} \lambda_{31} \lambda_{13} + \\
& + \frac{(N-2)(N^3+13N^2-33N+27)}{N^6} \lambda_{22}^2 + \\
& + \frac{(N-1)(N-2)(N+3)(N+15)}{N^5} \lambda_{22} \lambda_{11}^2 + \\
& + 4 \frac{(N-1)(N-2)}{N^5} \lambda_{30} \lambda_{03} \lambda_{11} + 4 \frac{(N-2)(2N^2+9N-27)}{N^5} \lambda_{21} \lambda_{12} \lambda_{11} + \\
& + 2 \frac{(N-2)(N+1)(N+3)}{N^4} \lambda_{11}^4.
\end{aligned}$$

.....

$$\begin{aligned}
D_{003} = & \frac{(N-1)(N-2)}{(N+1)} \left[\frac{(N-1)^2}{N^5} \lambda_{33} + 3 \frac{(N-1)(N+3)}{N^4} \lambda_{22} \lambda_{11} + \right. \\
& \left. + \frac{N+1}{N^4} \lambda_{30} \lambda_{03} + 3 \frac{N-3}{N^4} \lambda_{21} \lambda_{12} + \frac{(N+1)(N+3)}{N^3} \lambda_{11}^3 \right].
\end{aligned}$$

$$\begin{aligned}
D_{103} = & \frac{(N-1)(N-2)}{(N+1)} \left[\frac{(N-1)^3}{N^7} \lambda_{53} + 3 \frac{(N-1)^2(N+5)}{N^6} \lambda_{42} \lambda_{11} + \right. \\
& + \frac{(N-1)(N+1)}{N^6} \lambda_{50} \lambda_{03} + 3 \frac{(N-1)(3N-5)}{N^6} \lambda_{41} \lambda_{12} + \\
& + 15 \frac{(N-1)(N-3)}{N^6} \lambda_{32} \lambda_{21} + \frac{(N-1)(7N-5)}{N^6} \lambda_{23} \lambda_{30} + \\
& + 2 \frac{N-3}{N^5} \lambda_{40} \lambda_{13} + 3 \frac{N^3+7N^2-19N+15}{N^6} \lambda_{31} \lambda_{22} + \\
& \left. + 3 \frac{(N-3)(N+5)}{N^5} \lambda_{21}^2 \lambda_{11} + 3 \frac{(N-1)(N+3)(N+5)}{N^5} \lambda_{31} \lambda_{11}^2 \right].
\end{aligned}$$

(110)

.....

$$\begin{aligned}
D_{004} = & \frac{(N-1)(N-2)}{(N+1)(N+3)} \left[\frac{(N-1)^3}{N^6} \lambda_{44} + 4 \frac{(N-1)^2(N+5)}{N^5} \lambda_{33} \lambda_{11} + \right. \\
& + 4 \frac{(N-1)(N+1)}{N^5} (\lambda_{41} \lambda_{03} + \lambda_{14} \lambda_{30}) + \\
& + 12 \frac{(N-1)(N-3)}{N^5} (\lambda_{32} \lambda_{12} + \lambda_{23} \lambda_{21}) + \frac{(N-3)(N+3)}{N^5} \lambda_{40} \lambda_{04} + \\
& + 4 \frac{(N-3)^2}{N^5} \lambda_{31} \lambda_{13} + 3 \frac{(N-1)(N^2+7N-6)}{N^5} \lambda_{22}^2 + \\
& + 6 \frac{(N-1)(N+3)(N+5)}{N^4} \lambda_{22} \lambda_{11}^2 + 4 \frac{(N+1)(N+5)}{N^4} \lambda_{30} \lambda_{03} \lambda_{11} + \\
& \left. + 12 \frac{(N-3)(N+5)}{N^4} \lambda_{21} \lambda_{12} \lambda_{11} + \frac{(N+1)(N+3)(N+5)}{N^3} \lambda_{11}^4 \right].
\end{aligned}$$

In the event of complete independence we get

$$\begin{aligned}
 D_{ij1} &= 0 \text{ if } i+j \leq 3 \\
 D_{ij2} &= 0 \text{ if } i+j \leq 2 \\
 D_{003} &= \frac{(N-1)(N-2)}{N^4} \lambda_{30} \lambda_{03} \\
 D_{103} &= \frac{(N-1)^2(N-2)}{N^6} \lambda_{50} \lambda_{03} \\
 D_{004} &= \frac{(N-1)(N-2)(N-3)}{N^5(N+1)} \lambda_{40} \lambda_{04}.
 \end{aligned}
 \tag{111}$$

.....

The first expressions seem to indicate that all the terms of type D_{ij1} and $D_{ij2}=0$ if both the variates are completely independent. This can also be proved by the following more general deduction of the coefficients B_{lmnij} and C_{lmn} .

The coefficients B_{lmnij} are defined by the relation

$$\begin{aligned}
 1 + \sum \sum \sum \sum \sum \sum \frac{B_{lmnij}}{l! m! n! i! j!} t_1^l t_2^m t_3^n t_4^i t_5^j = \\
 = \left[1 + \sum \sum \sum \sum \sum \sum \frac{A_{2l+n+i, 2m+n+j}}{N^{l+m+n+i+j} l! m! n! i! j!} t_1^l t_2^m t_3^n t_4^i t_5^j \right]^N.
 \end{aligned}
 \tag{112}$$

Putting

$$1 + \sum \sum \frac{A_{ij}}{N^{i+j} i! j!} t_4^i t_5^j = u(t_4, t_5),$$

we obtain

$$\begin{aligned}
 1 + \sum \sum \sum \sum \sum \sum \frac{B_{lmnij}}{l! m! n! i! j!} t_1^l t_2^m t_3^n t_4^i t_5^j = \\
 = \left[u(t_4, t_5) + \sum \sum \sum \frac{N^{l+m+n}}{l! m! n!} \frac{\delta^{2l+2m} u}{\delta t_4^{2l} \delta t_5^{2m}} t_1^l t_2^m t_3^n \right]^N.
 \end{aligned}
 \tag{113}$$

Putting

$$u(t_4, t_5) + \sum \sum \frac{N^{l+m}}{l! m!} \frac{\delta^{2l+2m} u}{\delta t_4^{2l} \delta t_5^{2m}} t_1^l t_2^m = U(t_1, t_2, t_4, t_5),$$

the last expression will be

$$\left[U(t_1, t_2, t_4, t_5) + N t_3 \frac{\delta^2 U}{\delta t_4^2 \delta t_5^2} + \frac{N^2 t_3^2}{2!} \frac{\delta^4 U}{\delta t_4^2 \delta t_5^2} + \dots \right]^N.
 \tag{114}$$

We obtain

$$\begin{aligned}
 & \sum \frac{B_{lm0ij}}{l!m!i!j!} t_1^l t_2^m t_4^i t_5^j = U^N \\
 (115) \quad & \sum \frac{B_{lm1ij}}{l!m!i!j!} t_1^l t_2^m t_4^i t_5^j = N^2 U^{N-1} \frac{\delta^2 U}{\delta t_4 \delta t_5} \\
 & \sum \frac{B_{lm2ij}}{l!m!i!j!} t_1^l t_2^m t_4^i t_5^j = N^3 U^{N-1} \frac{\delta^4 U}{\delta t_4^2 \delta t_5^2} + N^3 (N-1) U^{N-2} \left(\frac{\delta^2 U}{\delta t_4 \delta t_5} \right)^2; \\
 & \vdots
 \end{aligned}$$

As an intermediate link between the B -coefficients and the C -coefficients the following coefficients B'_{lmnij} may be inserted.

$$(116) \quad B'_{lmnij} = B_{lmnij} - n B_{lm, n-1, i+1, j+1} + \dots + (-1)^n B_{lm0i+n, j+n}.$$

We obtain

$$\begin{aligned}
 & \sum \frac{B'_{lm0ij}}{l!m!i!j!} t_1^l t_2^m t_4^i t_5^j = U^N \\
 & \sum \frac{B'_{lm1ij}}{l!m!i!j!} t_1^l t_2^m t_4^i t_5^j = U^N N(N-1) f_{11} \\
 & \sum \frac{B'_{lm2ij}}{l!m!i!j!} t_1^l t_2^m t_4^i t_5^j = U^N N(N-1) [(N-1) f_{22} + 2 N^2 f_{11}^2 + N f_{20} f_{02}] \\
 (117) \quad & \sum \frac{B'_{lm3ij}}{l!m!i!j!} t_1^l t_2^m t_4^i t_5^j = U^N N(N-1) [N(N-2) f_{30} f_{03} + \\
 & \quad + \text{other terms involving mixed derivatives of } f] \\
 & \sum \frac{B'_{lm4ij}}{l!m!i!j!} t_1^l t_2^m t_4^i t_5^j = U^N N(N-1) [(N^3 - 3N^2 + 3N) f_{40} f_{04} + \\
 & \quad + 3N^2 (N-1) (f_{40} f_{02}^2 + f_{04} f_{20}^2) + 3N^3 (N+1) f_{20}^2 f_{02}^2 + \\
 & \quad + \text{other terms involving mixed derivatives of } f].
 \end{aligned}$$

In these expressions

$$f = \log U \quad \text{and} \quad f_{ij} = \frac{\delta^{i+j} f}{\delta t_4^i \delta t_5^j}.$$

If the variates of the parent population are completely independent, we have

$$u(t_4, t_5) = u_1(t_4) \cdot u_2(t_5)$$

and

$$\begin{aligned} U(t_1, t_2, t_4, t_5) &= \left[u_1 + \sum \frac{N^n t_1^n}{n!} \frac{\delta^{2n} u_1}{\delta t_4^{2n}} \right] \left[u_2 + \sum \frac{N^n t_2^n}{n!} \frac{\delta^{2n} u_2}{\delta t_5^{2n}} \right] = \\ &= U_1(t_1, t_4) \cdot U_2(t_2, t_5), \end{aligned}$$

from which it immediately follows that

$$f_{ij} = 0 \text{ if both } i \text{ and } j \neq 0;$$

$$f_{i0} = \frac{\delta^i}{\delta t_4^i} \log U_1;$$

$$f_{0j} = \frac{\delta^j}{\delta t_5^j} \log U_2.$$

Hence in this case

$$B'_{lm1ij} = 0.$$

From these inserted intermediate B'_{lmnij} coefficients we obtain C_{lmn} by the relation

$$C_{lmn} = B'_{lmn00} - l B'_{l-1,mn20} - m B'_{l,m-1,n02} + \dots + (-1)^{l+m} B'_{00n2l,2m}.$$

It immediately follows that $C_{lm1} = 0$ if the variates are independent.

For the lowest values of n the coefficients can be given by the following relations

$$\begin{aligned} \sum \sum \frac{C_{lm0}}{l!m!} t_1^l t_2^m &= \left[U^N + \dots (-1)^{i+j} \frac{t_1^i t_2^j}{i!j!} \frac{\delta^{2i+2j}}{\delta t_4^{2i} \delta t_5^{2j}} U^N \dots \right]_{t_4=t_5=0} \\ \sum \sum \frac{C_{lm1}}{l!m!} t_1^l t_2^m &= N(N-1) \cdot \\ &\cdot \left[U^N f_{11} + \dots (-1)^{i+j} \frac{t_1^i t_2^j}{i!j!} \frac{\delta^{2i+2j}}{\delta t_4^{2i} \delta t_5^{2j}} (U^N f_{11}) \dots \right]_{t_4=t_5=0} \\ (118) \quad \sum \sum \frac{C_{lm2}}{l!m!} t_1^l t_2^m &= \\ &= N(N-1)^2 \left[U^N f_{22} + \dots (-1)^{i+j} \frac{t_1^i t_2^j}{i!j!} \frac{\delta^{2i+2j}}{\delta t_4^{2i} \delta t_5^{2j}} (U^N f_{22}) \dots \right]_{t_4=t_5=0} + \\ &+ 2N^3(N-1) \left[U^N f_{11}^2 + \dots (-1)^{i+j} \frac{t_1^i t_2^j}{i!j!} \frac{\delta^{2i+2j}}{\delta t_4^{2i} \delta t_5^{2j}} (U^N f_{11}^2) \dots \right]_{t_4=t_5=0} + \\ &+ N^2(N-1) \left[U^N f_{20} f_{02} + \dots (-1)^{i+j} \frac{t_1^i t_2^j}{i!j!} \frac{\delta^{2i+2j}}{\delta t_4^{2i} \delta t_5^{2j}} (U^N f_{20} f_{02}) \dots \right]_{t_4=t_5=0}. \end{aligned}$$

The coefficients D_{lm2} are given by the formula

$$D_{lm2} = C_{lm2} - \frac{1}{N-1} C_{l+1, m+1, 0}$$

or in general

$$(119) \quad \sum \sum \frac{D_{lm2}}{l!m!} t_1^l t_2^m = \sum \sum \frac{C_{lm2}}{l!m!} t_1^l t_2^m - \frac{1}{N-1} \frac{\delta^2}{\delta t_1 \delta t_2} \sum \sum \frac{C_{lm0}}{l!m!} t_1^l t_2^m.$$

But

$$\begin{aligned} & \frac{\delta^2}{\delta t_1 \delta t_2} \sum \sum \frac{C_{lm0}}{l!m!} t_1^l t_2^m = \\ & = \left[\left(\frac{\delta^2}{\delta t_1 \delta t_2} U^N - \frac{\delta}{\delta t_2} \frac{\delta^2}{\delta t_1^2} U^N - \frac{\delta}{\delta t_1} \frac{\delta^2}{\delta t_2^2} U^N + \frac{\delta^4}{\delta t_1^2 \delta t_2^2} U^N \right) + \right. \\ & \quad \left. + \dots (-1)^{i+j} \frac{t_1^i t_2^j}{i!j!} \frac{\delta^{2i+2j}}{\delta t_1^{2i} \delta t_2^{2j}} \left(\frac{\delta^2}{\delta t_1 \delta t_2} U^N - \frac{\delta}{\delta t_2} \frac{\delta^2}{\delta t_1^2} U^N - \frac{\delta}{\delta t_1} \frac{\delta^2}{\delta t_2^2} U^N + \right. \right. \\ & \quad \left. \left. + \frac{\delta^4}{\delta t_1^2 \delta t_2^2} U^N \right) \dots \right]_{t_4=t_5=0}. \end{aligned}$$

Since $\frac{\delta}{\delta t_1} U = N \cdot \frac{\delta}{\delta t_1^2} U$ the expression inside the parenthesis becomes

$$U^N [N(N-1)^2 f_{22} + N^2(N-1)^2 f_{20} f_{02} + 2N^2(N-1) f_{11}^2].$$

From this it follows that

$$\begin{aligned} & \sum \sum \frac{D_{lm2}}{l!m!} t_1^l t_2^m = \\ & = N(N-1)(N-2) \left[U^N f_{22} + \dots (-1)^{i+j} \frac{t_1^i t_2^j}{i!j!} \frac{\delta^{2i+2j}}{\delta t_1^{2i} \delta t_2^{2j}} (U^N f_{22}) + \dots \right]_{t_4=t_5=0} + \\ (120) \quad & + 2N^2(N^2 - N - 1) \left[U^N f_{11}^2 + \dots (-1)^{i+j} \frac{t_1^i t_2^j}{i!j!} \frac{\delta^{2i+2j}}{\delta t_1^{2i} \delta t_2^{2j}} (U^N f_{11}^2) + \dots \right]_{t_4=t_5=0}. \end{aligned}$$

When the variates are independent both f_{22} and f_{11} are zero and therefore $D_{lm2} = 0$.

When the variates are independent we also get

$$\begin{aligned} & \sum \sum \frac{D_{lm3}}{l!m!} t_1^l t_2^m = N^2(N-1)(N-2) \cdot \\ (121) \quad & \left[U_1^N f_{30} + \dots (-1)^i \frac{t_1^i}{i!} \frac{\delta^{2i}}{\delta t_1^{2i}} (U_1^N f_{30}) \dots \right]_{t_4=0} \left[U_2^N f_{03} + \right. \\ & \quad \left. + \dots (-1)^j \frac{t_2^j}{j!} \frac{\delta^{2j}}{\delta t_2^{2j}} (U_2^N f_{03}) \dots \right]_{t_5=0} \end{aligned}$$

$$\begin{aligned}
 \sum_l \sum_m \frac{D_{lm4}}{l!m!} t_1^l t_2^m &= \frac{N^3(N-1)(N-2)(N-3)}{(N+1)} \\
 (121) \quad &\cdot \left[U_1^N f_{40} + \dots + 1) \frac{t_1^i}{i!} \frac{\partial^{2i}}{\partial t_1^{2i}} (U_1^N f_{40}) \dots \right]_{t_1=0} \left[U_2^N f_{04} + \dots \right. \\
 &\quad \left. \dots + 1) \frac{t_2^j}{j!} \frac{\partial^{2j}}{\partial t_2^{2j}} (U_2^N f_{04}) \dots \right]_{t_2=0}.
 \end{aligned}$$

These formulas suggest that the higher coefficients may also be given by similar simple expressions.

As a further special case mention may be made of that in which one of the variates is normally distributed and complete independence exists.

In this case $A_{ij} = 0$ if $j \neq 0$.

All the B coefficients except B_{i0j0} vanish and consequently also the C -coefficients except C_{i00} .

In such a case we get

$$F(r_{20}, r_{02}, r_{11}) = F_0(r_{20}, r_{02}, r_{11}) \left[1 + \sum \frac{C_{i00}}{i!} \frac{N-1}{2} L_i(r_{20}) \right].$$

CHAPTER 12.

The Distribution and the Moments of the Correlation Coefficient.

From the distribution obtained in the preceding chapter of the three moments of the second order, the distribution of the correlation coefficient is obtained without any considerable difficulty

$$(122) \quad F(\nu_{20}, \nu_{02}, \varrho) = F(\nu_{20}, \nu_{02}, \varrho \nu_{20}^{1/2} \nu_{02}^{1/2}) \nu_{20}^{1/2} \nu_{02}^{1/2}$$

and

$$(123) \quad F(\varrho) = \int_0^\infty \int_0^\infty F(\nu_{20}, \nu_{02}, \varrho) d\nu_{20} d\nu_{02}.$$

The term

$$F_0(\nu_{20}, \nu_{02}, \nu_{11}) = C_0 [\nu_{20} \nu_{02} - \nu_{11}^2]^{\frac{N-4}{2}} e^{-\frac{N}{2}(\nu_{20} + \nu_{02})}$$

becomes

$$F_0(\nu_{20}, \nu_{02}, \varrho) = C_0 [1 - \varrho^2]^{\frac{N-4}{2}} \nu_{20}^{\frac{N-3}{2}} \nu_{02}^{\frac{N-3}{2}} e^{-\frac{N}{2}(\nu_{20} + \nu_{02})}$$

or

$$(124) \quad F_0(\varrho) = C_0' [1 - \varrho^2]^{\frac{N-4}{2}}; \quad C_0' = C_0 \frac{\left(\Gamma\left(\frac{N-1}{2}\right)\right)^2}{\left(\frac{N}{2}\right)^{N-1}},$$

or the formula found by Student and R. A. Fisher. [The expression $C_0'[1 - \varrho^2]^{\frac{N-4}{2}}$ is denoted in the following exposition by $\psi_0(\varrho)$ and the expression $C_n'[1 - \varrho^2]^{\frac{N-4}{2}+n}$ by $\psi_n(\varrho)$.]

The expression

$$e^{-\frac{N}{2}(\nu_{20} + \nu_{02})} \frac{\delta^n}{\delta \nu_{11}^n} \psi_n(\nu_{20}, \nu_{02}, \nu_{11})^{\frac{N-1}{2}+n} L_i(\nu_{20})^{\frac{N-1}{2}+n} L_j(\nu_{02}),$$

since

$$\frac{\delta}{\delta v_{11}} = \frac{\delta}{\delta \varrho} \cdot \frac{1}{\sqrt{v_{20} v_{02}}},$$

becomes

$$(-1)^{i+j} \frac{\delta^n}{\delta \varrho^n} \psi_n(\varrho) \left[\frac{1}{v_{20} v_{02}} \right]^{n/2} f_{\frac{N-1}{2}+n+i}^{(i)}(v_{20}) f_{\frac{N-1}{2}+n+j}^{(j)}(v_{02}).$$

But

$$\begin{aligned} & \int_0^\infty \frac{1}{v_{20}^{\frac{n}{2}+i}} f_{\frac{N-1}{2}+n+i}^{(i)}(v_{20}) dv_{20} = \\ & = \frac{n}{2} \cdot \left(\frac{n}{2} + 1\right) \left(\frac{n}{2} + 2\right) \cdots \left(\frac{n}{2} + i - 1\right) \int_0^\infty \frac{1}{v_{20}^{\frac{n}{2}+i}} f_{\frac{N-1}{2}+n+i}^{(i)}(v_{20}) = \\ & = \frac{n}{2} \cdot \left(\frac{n}{2} + 1\right) \cdots \left(\frac{n}{2} + i - 1\right) \cdot \frac{\left(\frac{N}{2}\right)^{\frac{n}{2}+i} \Gamma\left(\frac{N-1}{2} + \frac{n}{2}\right)}{\Gamma\left(\frac{N-1}{2} + n + i\right)} = \end{aligned}$$

For $n=0$ the expression becomes 0 if $i \neq 0$.

We accordingly obtain

$$(125) \quad F(\varrho) = \psi_0(\varrho) - E_1 \cdot \frac{\delta}{\delta \varrho} \psi_1(\varrho) + \frac{1}{2!} E_2 \frac{\delta^2}{\delta \varrho^2} \psi_2(\varrho) - \dots,$$

i. e. a »Romanowsky's Generalization of Type II«.

For the different coefficients we have

$$\begin{aligned} (126) \quad E_1 &= \frac{N}{2} \left[\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N+1}{2}\right)} \right]^2 \left[D_{001} - \frac{1}{2} \frac{N}{N+1} (D_{101} + D_{011}) + \right. \\ &+ \frac{1}{2!} \frac{1}{4} \left(3 \frac{N^2}{(N+1)(N+3)} (D_{201} + D_{021}) + 2 \frac{N^2}{(N+1)^2} D_{111} \right) - \\ &- \frac{1}{3!} \frac{1}{8} \left(15 \frac{N^3}{(N+1)(N+3)(N+5)} (D_{301} + D_{031}) + \right. \\ &\left. \left. + 9 \frac{N^3}{(N+1)^2(N+3)} (D_{211} + D_{121}) \right) + \dots \right]. \\ E_2 &= \left(\frac{N}{N+1} \right)^2 \left[D_{002} - \frac{N}{N+3} (D_{102} + D_{012}) + \right. \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2!} \left(2 \frac{N^2}{(N+3)(N+5)} (D_{202} + D_{022}) + 2 \frac{N^2}{(N+3)^2} D_{112} \right) - \dots \Big]. \\
 (126) \quad E_3 &= \binom{N}{2}^3 \left[\frac{\Gamma\left(\frac{N+2}{2}\right)}{\Gamma\left(\frac{N+5}{2}\right)} \right]^2 \left[D_{003} - \frac{3}{2} \frac{N}{N+5} (D_{103} + D_{013}) + \dots \right]. \\
 E_4 &= \frac{N^4}{(N+3)^2(N+5)^2} [D_{004} - \dots].
 \end{aligned}$$

If terms of the order $\frac{1}{N^2}$ are neglected, we get

$$\begin{aligned}
 E_1 &= \frac{2N-1}{2N} r - \frac{1}{2N} (\gamma_{31} + \gamma_{13}) + \frac{r}{2N} \left[\frac{3}{4} (\gamma_{40} + \gamma_{04}) + \frac{2}{4} \gamma_{22} + r^2 \right]. \\
 E_2 &= \left(1 - \frac{3}{N} \right) r^2 + \frac{1}{N} \gamma_{22} - \frac{2}{N} r (\gamma_{31} + \gamma_{13}) + \frac{r^2}{N} [\gamma_{40} + \gamma_{04} + \gamma_{22} + 2r^2].
 \end{aligned}$$

[If terms of the order, $\frac{1}{N}$ are to be included, the expression for E_3 will come to contain the coefficients D_{203} , D_{113} , and D_{023} , that is terms of the fifth order, and the expression for E_4 terms of the sixth order.]

Hence we find the following values of the first two moments of the correlation coefficient, if terms of order $\frac{1}{N^2}$ are neglected,

$$(127 \text{ a}) \quad m(\varrho) = r - \frac{1}{2N} [r + \gamma_{31} + \gamma_{13}] + \frac{r}{8N} [4r^2 + 3(\gamma_{40} + \gamma_{04}) + 2\gamma_{22}].$$

$$(127 \text{ b}) \quad \sigma^2(\varrho) = \frac{1}{N} [1 - r^2]^2 + \frac{1}{N} [\gamma_{22} - r(\gamma_{31} + \gamma_{13}) + \frac{r^2}{4} (\gamma_{40} + \gamma_{04} + 2\gamma_{22})].$$

This formula (127 b) is in accordance with the formula already obtained by Sheppard¹ except that instead of the standardized moments the standardized seminvariants of the parent population have been used.

As is known, the different derivatives $\frac{\delta^n \psi_n(\varrho)}{\delta \varrho^n}$ can be written as a product of $\psi_0(\varrho)$ and a polynomial of the n th degree $P_n(\varrho)$; $\frac{\delta^n \psi_n}{\delta \varrho^n} = (-1)^n \psi_0(\varrho) P_n(\varrho)$.

¹ W. F. SHEPPARD. »On the Application of the Theory of Errors to Cases of Normal Distributions and Normal Correlation«. Phil. Trans. A. 192. (1899). Compare also »On the probable Error of Frequency Constants«. (Editorial) Biometrika 9. 1913.

We find that

$$\begin{aligned}
 P_1(\varrho) &= (N-1)\varrho \\
 P_2(\varrho) &= \frac{(N-1)(N+1)}{N-2} [(N-1)\varrho^2 - 1] \\
 (128 \text{ a}) \quad P_3(\varrho) &= \frac{(N-1)(N+1)(N+3)}{(N-2)} [(N+1)\varrho^3 - 3\varrho] \\
 P_4(\varrho) &= \frac{(N-1)(N+1)(N+3)(N+5)}{(N-2)N} [(N+1)(N+3)\varrho^4 - 6(N+1)\varrho^2 + 3]
 \end{aligned}$$

and in general

$$\begin{aligned}
 P_{2n}(\varrho) &= \frac{(N-1)(N+1)\dots(N+4n-3)}{(N-2)N\dots(N+2n-4)} \left[(N+2n-3)(N+2n-1)\dots \right. \\
 &\quad \left. \dots (N+4n-5)\varrho^{2n} - \frac{2n(2n-1)}{2 \cdot 1!} (N+2n-3)\dots(N+4n-7)\varrho^{2n-2}\dots \right. \\
 &\quad \left. \dots (-1)^n \frac{2n!}{2^n n!} \right] \\
 (128 \text{ b}) \quad P_{2n+1}(\varrho) &= \frac{(N-1)(N+1)\dots(N+4n-1)}{(N-2)N\dots(N+2n-4)} \left[(N+2n-1)(N+2n+1)\dots \right. \\
 &\quad \left. \dots (N+4n-3)\varrho^{2n+1} - \frac{(2n+1)2n}{2 \cdot 1!} (N+2n-1)\dots(N+4n-5)\varrho^{2n-1}\dots \right. \\
 &\quad \left. \dots (-1)^n \frac{(2n+1)!}{2^n n!} \varrho \right].
 \end{aligned}$$

If N is large so that terms of the order $\frac{1}{N}$ can be neglected, the polynomials become the ordinary Hermitian polynomials, where $\lambda_2 = \frac{1}{N}$, just as the frequency function $\psi_0(\varrho)$ becomes the normal frequency curve.

If ϱ is of the order $\frac{1}{\sqrt{N}}$, the polynomials assume the order $N^{\frac{n}{2}}$. If ϱ is of higher order the polynomials assume the order N^n .

For the given development in series to be of practical use the value of the different terms must decrease with increasing powers. It is true that according to Romanowsky a series development like the one given, when the variate must be situated between certain limits, should be convergent, but the practical application of a series of this type demands that the value of the several terms should rapidly diminish. When the polynomials increase by the order $N^{\frac{n}{2}}$ (mini-

mum), it is necessary for the E coefficients to diminish more rapidly than $N^{-\frac{n}{2}}$. However, this is not possible unless r is zero or of a lower order than $\frac{1}{\sqrt{N}}$, i.e. of a lower order than its mean error (i.e. practically $= 0$).

This law of distribution can accordingly be used only in those cases in which the first correlation moment is equal to zero, and hence it shows how the higher characteristics of the parent population modify the law of distribution found by »Student».

Hence, *if the parent population can be described by means of a »Charlier's Correlation Surface of Type A», and if the correlation coefficient of the parent population is zero, the distribution of the correlation coefficient in random samples can be described by means of a »Romanowsky's Generalization of Type II».*

The following special cases may be mentioned.

I. Complete independent variates.

As shown before, D_{ij1} as well as D_{ij2} are identically equal to zero if the variates are independent. Hence the mean value of the correlation coefficient is zero and its standard deviation is the same as in a normally uncorrelated parent population, that is $= \frac{1}{\sqrt{N-1}}$.

II. Complete independent variates and one of the variates symmetrically distributed

(i.e. the odd seminvariants are all equal to zero). In that case all the $C_{ij,2n+1}$ coefficients will be equal to zero; consequently $E_{2n+1} = 0$. The distribution of the correlation coefficient will then be symmetrical.

III. Complete independent variates and one of the variates normally distributed.

In that case all the C -coefficients except C_{i00} (or C_{0i0}) will be identically equal to zero, and in consequence all the E coefficients $= 0$.

The law of distribution deduced by »Student» and R. A. Fisher for correlation coefficients in random samples of a normal uncorrelated distribution will accordingly also apply to the more extensive case of only one of the variates being normally distributed, while the distribution of the second variate may be any distribution whatever. Properly, according to the deduction given here, the distribution of the variate is to be described by means of a »Charlier's A-Series», but as theoretically most distributions can be described by means of such a series this restriction is rather unimportant.

CHAPTER 13.

The Mean Value and the Mean Error of the Regression Coefficient.

The regression coefficient is given by the formula $\frac{\nu_{11}}{\nu_{20}}$. Although I shall not here study the distribution of this coefficient, I shall deduce the two first moments of this distribution, as it will be easy to derive the moments of the distribution of these coefficients from the characteristic function.

If the characteristic function of the distribution of ν_{11} and ν_{20} is $U(t_1, t_3)$, the mean value of ν_{11}^n is determinable as a function of ν_{20} by the relation

$$(129) \quad m_{\nu_{20}}(\nu_{11}^n) = \frac{1}{f(\nu_{20})} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\nu_{20}\omega_1 i} \frac{\delta^n}{\delta t_3^n} [U(\omega_1 i, t_3)]_{t_3=0} d\omega_1$$

and we obtain

$$(130) \quad m \left(\frac{\nu_{11}}{\nu_{20}} \right)^n = \int_0^{\infty} \frac{d\nu_{20}}{\nu_{20}^n} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\nu_{20}\omega_1 i} \frac{\delta^n}{\delta t_3^n} [U(\omega_1 i, t_3)]_{t_3=0} d\omega_1.$$

If the special formula, where $r=0$, is used, the characteristic function $U(t_1, t_3)$ is given by the following equation obtained from formula (92)

$$(131) \quad U(t_1, t_3) = \left[1 - \frac{2\lambda_{20}}{N} t_1 - \frac{\lambda_{20}\lambda_{02} t_3^2}{N^2} \right]^{-\frac{N-1}{2}} \left[1 + \sum \sum \sum \frac{C_{ijk}}{i! j! k!} \tau_1^i \tau_2^j \tau_3^k \right],$$

where according to formula (82 b)

$$\tau_1 = \frac{t_1 + \frac{\lambda_{02} t_3^2}{2N}}{1 - 2 \frac{\lambda_{20}}{N} t_1 - \frac{\lambda_{20}\lambda_{02}}{N^2} t_3^2};$$

$$\tau_2 = \frac{\frac{\lambda_{20} t_3^2}{2N}}{1 - \frac{2\lambda_{20}}{N} t_1 - \frac{\lambda_{20} \lambda_{02}}{N^2} t_3^2};$$

$$\tau_3 = \frac{t_3}{1 - \frac{2\lambda_{20}}{N} t_1 - \frac{\lambda_{20} \lambda_{02}}{N^2} t_3^2}.$$

(In these formulas the seminvariants of the second order, λ_{20} and λ_{02} , have been retained in the expressions for τ_1 , τ_2 and τ_3 , while λ_{11} is included in the coefficients C_{ijk} , which are given in formula 94.)

We obtain

$$(132 \text{ a}) \quad \left[\frac{\delta U}{\delta t_3} \right]_{t_3=0} = \left[1 - \frac{2\lambda_{20}}{N} t_1 \right]^{-\frac{N+1}{2}} \sum \frac{C_{i01}}{i!} \tau_1^i$$

and

$$(132 \text{ b}) \quad \left[\frac{\delta^2 U}{\delta t_3^2} \right]_{t_3=0} = \frac{N-1}{N^2} \lambda_{20} \lambda_{02} \left[1 - \frac{2\lambda_{20}}{N} t_1 \right]^{-\frac{N+1}{2}} \left[1 + \sum \frac{C_{i00}}{i!} \tau_1^i \right] +$$

$$+ \frac{\lambda_{20}}{N} \left[1 - \frac{2\lambda_{20}}{N} t_1 \right]^{-\frac{N+1}{2}} \sum \frac{C_{i10}}{i!} \tau_1^i +$$

$$+ \frac{\lambda_{02}}{N} \left[1 - \frac{2\lambda_{20}}{N} t_1 \right]^{-\frac{N+3}{2}} \sum \frac{C_{i+1,00}}{i!} \tau_1^i +$$

$$+ \left[1 - \frac{2\lambda_{20}}{N} t_1 \right]^{-\frac{N+3}{2}} \sum \frac{C_{i,02}}{i!} \tau_1^i,$$

where now

$$\tau_1 = \frac{t_1}{1 - \frac{2\lambda_{20}}{N} t_1}.$$

Since

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-v_{20} \omega_1 i} \left[1 - \frac{2\lambda_{20}}{N} \omega_1 i \right]^{-\frac{N+1}{2} - n} (\omega_1 i)^n d\omega_1 =$$

$$= (-1)^n f_{\frac{N+1}{2} + n}^{(n)}(v_{20}); \quad f_{\frac{N+1}{2} + n}(v_{20}) = \left(\frac{N}{2\lambda_{20}} \right)^{\frac{N+1}{2} + n} \cdot \frac{v_{20}^{\frac{N+1}{2} + n - 1}}{\Gamma\left(\frac{N+1}{2} + n\right)} e^{-\frac{Nv_{20}}{2\lambda_{20}}};$$

and as

$$\begin{aligned} & \int_0^{\infty} \frac{1}{v_{20}} f_{\frac{N+1}{2}+n}^{(n)}(v_{20}) dv_{20} = n! \int_0^{\infty} \frac{1}{v_{20}^{n+1}} f_{\frac{N+1}{2}+n}^{(n)}(v_{20}) dv_{20} = \\ & = n! \left(\frac{N}{2\lambda_{20}} \right)^{n+1} \frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N+1}{2}+n\right)} = n! \frac{N^{n+1}}{\lambda_{20}^{n+1} (N-1)(N+1)(N+3) \dots (N+2n-1)}, \end{aligned}$$

then

(133)

$$m\left(\frac{v_{11}}{v_{20}}\right) = \frac{C_{001}}{\lambda_{20}} \cdot \frac{N}{(N-1)} - \frac{C_{101}}{(\lambda_{20})^2} \cdot \frac{N^2}{(N-1)(N+1)} + \frac{C_{201}}{(\lambda_{20})^3} \cdot \frac{N^3}{(N-1)(N+1)(N+3)} - \dots$$

If only terms of the magnitude $\frac{1}{N}$ are included, then

$$\begin{aligned} (133 \text{ a}) \quad m\left(\frac{v_{11}}{v_{20}}\right) &= \frac{\lambda_{11}}{\lambda_{20}} - \frac{N-1}{N(N+1)} \left[\frac{\lambda_{31}\lambda_{20} - \lambda_{40}\lambda_{11}}{\lambda_{20}^3} \right] + \\ &+ \text{terms of order } \frac{1}{N^2}. \end{aligned}$$

If the regression is strictly linear, then according to Wicksell $\lambda_{k,1}\lambda_{20} = \lambda_{k+1,0}\lambda_{11}$, in which case $m\left(\frac{v_{11}}{v_{20}}\right) = \frac{\lambda_{11}}{\lambda_{20}} + \text{terms of order } \frac{1}{N^2}$.¹

If the regression is not strictly linear but deviates only slightly from the linear type, the difference $\lambda_{31}\lambda_{20} - \lambda_{40}\lambda_{11}$ will be of a fairly low order.

In such a case the correction terms are of little importance.

If the two variates are completely independent, then $C_{i01} = 0$ and hence it follows that

$$m\left(\frac{v_{11}}{v_{20}}\right) = 0.$$

If the independent variate is normally distributed,

$$C_{n01} = \frac{(N-1)^{n+1}}{N^{2n+1}} \lambda_{2n+1,1}$$

and the mean value of the regression coefficients will be

$$(133 \text{ b}) \quad m\left(\frac{v_{11}}{v_{20}}\right) = \frac{\lambda_{11}}{\lambda_{20}} - \frac{N-1}{N(N+1)} \frac{\lambda_{31}}{\lambda_{20}^2} + \frac{(N-1)^2}{N^2(N+1)(N+3)} \frac{\lambda_{51}}{\lambda_{20}^3} \dots$$

¹ It seems as if all terms of lower order than $\frac{1}{N}$ will vanish so that the relation $m\left(\frac{v_{11}}{v_{20}}\right) = \frac{\lambda_{11}}{\lambda_{20}}$ is exact in the case of strictly linear regression.

In like manner we find

$$\begin{aligned}
 m\left(\frac{\nu_{11}}{\nu_{20}}\right)^2 &= \frac{\lambda_{02}}{\lambda_{20}} \frac{N}{(N-3)} \left[1 + \sum (-1)^n (n+1) \frac{C_{n00}}{\lambda_{20}^n} \frac{N^n}{(N+1)(N+3)\dots(N+2n-1)} \right] + \\
 &+ \frac{\lambda_{02}}{\lambda_{20}} \cdot \frac{N}{(N-1)(N-3)} \sum (-1)^n (n+1) \frac{C_{n10}}{\lambda_{20}^n \lambda_{02}} \frac{N^n}{(N+1)\dots(N+2n-1)} + \\
 (134) \quad &+ \frac{\lambda_{02}}{\lambda_{20}} \frac{N}{(N-1)(N+1)} \sum (-1)^n (n+1) \frac{C_{n+1,00}}{\lambda_{20}^{n+1}} \frac{N^n}{(N+3)\dots(N+2n+1)} + \\
 &+ \frac{\lambda_{02}}{\lambda_{20}} \frac{N^2}{(N-1)(N+1)} \sum (-1)^n (n+1) \frac{C_{n02}}{\lambda_{20}^{n+1} \lambda_{02}} \frac{N^n}{(N+3)\dots(N+2n+1)}
 \end{aligned}$$

or, if only terms of order $\frac{1}{N}$ are included,

$$\begin{aligned}
 m\left(\frac{\nu_{11}}{\nu_{20}}\right)^2 &= \frac{N}{N+1} \frac{\lambda_{11}^2}{\lambda_{20}^2} + \frac{1}{N-3} \cdot \frac{\lambda_{02}}{\lambda_{20}} + \frac{N-1}{N(N+1)} \frac{\lambda_{22}}{\lambda_{20}^2} - \\
 (134 \text{ a}) \quad &- 4 \frac{(N-1)(N+2)}{N(N+1)(N+3)} \frac{\lambda_{31} \lambda_{11}}{\lambda_{20}^3} + 3 \frac{(N-1)(N+4)}{N(N+1)(N+5)} \frac{\lambda_{40} \lambda_{11}^2}{\lambda_{20}^4}; \\
 \varepsilon^2\left(\frac{\nu_{11}}{\nu_{20}}\right) &= m\left(\frac{\nu_{11}}{\nu_{20}}\right)^2 - m^2\left(\frac{\nu_{11}}{\nu_{20}}\right) = \frac{1}{N-3} \cdot \frac{\lambda_{02}}{\lambda_{20}} - \frac{1}{N+1} \frac{\lambda_{11}^2}{\lambda_{20}^2} + \\
 &+ \frac{N-1}{N(N+1)} \frac{\lambda_{22}}{\lambda_{20}^2} - 2 \frac{(N-1)}{N(N+3)} \frac{\lambda_{31} \lambda_{11}}{\lambda_{20}^3} + \frac{(N-1)(N+2)}{N(N+1)(N+5)} \frac{\lambda_{40} \lambda_{11}^2}{\lambda_{20}^4}
 \end{aligned}$$

or approximately the wellknown formula

$$(135) \quad \varepsilon^2\left(\frac{\nu_{11}}{\nu_{20}}\right) = \frac{\lambda_{02}}{\lambda_{20}} \cdot \frac{1}{N} [1 - r^2 + \gamma_{22} - 2r\gamma_{31} + \gamma_{40}r^2].$$

If the two variates are uncorrelated, $C_{n10} = 0$, and it then also follows that

$$C_{n02} = \frac{C_{n-1,10}}{N-1} = 0.$$

If in addition the x -variate is normally distributed, $C_{n00} = 0$ and hence we find that the wellknown formula

$$\varepsilon^2\left(\frac{\nu_{11}}{\nu_{20}}\right) = \frac{\lambda_{02}}{\lambda_{20}} \cdot \frac{1}{N-3}$$

for the mean error of the regression coefficient applies, not only to the case in which both the variates are independent and normally distributed, but also to the case in which only the x -variate is normally distributed.

If this last condition is not fulfilled, but both the variates are independent, we have

$$\begin{aligned}
 \varepsilon^2 \left(\frac{\nu_{11}}{\nu_{20}} \right) &= \frac{\lambda_{02}}{\lambda_{20}} \cdot \frac{1}{N-3} + \\
 &+ \frac{\lambda_{02}}{\lambda_{20}} \left[\frac{1}{N-3} \sum_{n=1} (-1)^n (n+1) \frac{C_{n00}}{\lambda_{20}^n} \frac{N^n}{(N+1)(N+3) \dots (N+2n-1)} - \right. \\
 (135 \text{ b}) \quad &\left. - \frac{1}{N-1} \sum_{n=1} (-1)^n n \frac{C_{n00}}{\lambda_{20}^n} \frac{N^n}{(N+1) \dots (N+2n-1)} \right] = \\
 &= \frac{\lambda_{02}}{\lambda_{20}} \cdot \frac{1}{N-3} \left[1 + \sum_1 (-1)^n \frac{C_{n00}}{\lambda_{20}^n} \frac{N^n}{(N-1) \dots (N+2n-3)} \right]
 \end{aligned}$$

a rather simple expression.

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